Needham, Chapter 7, Exercise 18, p. 374
G. Palmer, June 1, 2017, 9:30 am PST

Out[3617]=

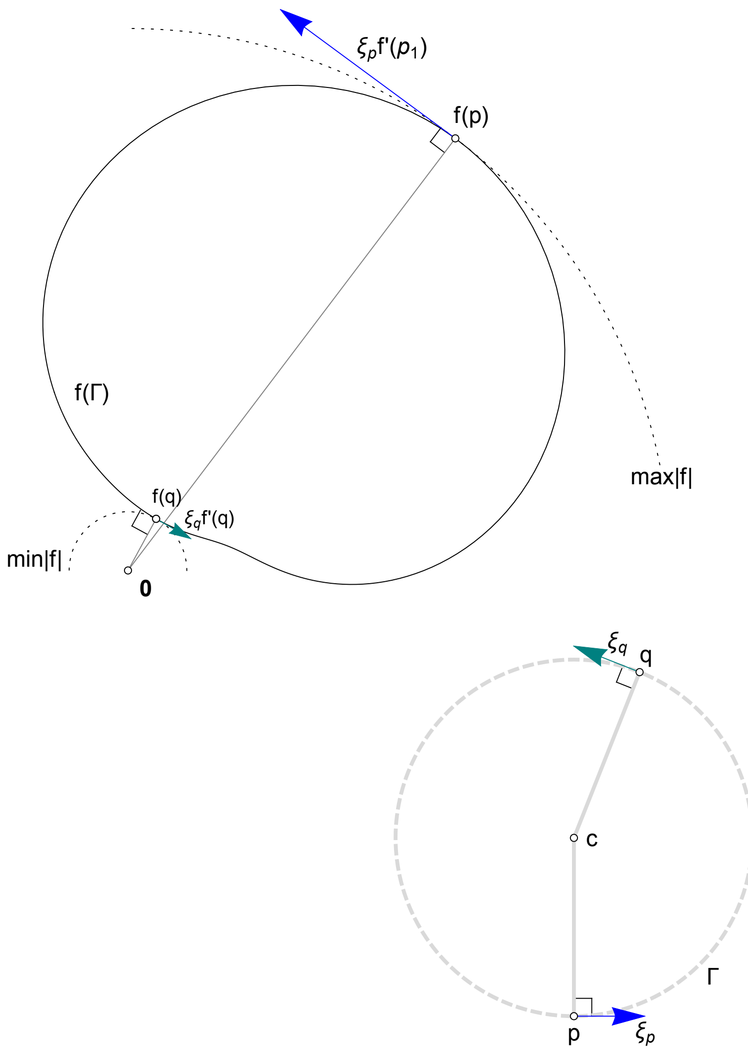

Figure 1: Direction $\xi f'(p)$ for $|f|_{max}$ and $|f|_{min}$
$f(z) = .5 \sin(z)$

18 Let f be analytic on a simple loop Γ, and let p be a preimage of a point on f(Γ) at which |f| is maximum. If ξ is a tangent complex number to Γ at p, and in the same counterclockwise sense as Γ, show geometrically that ξf′(p) points in the same direction as if(p). What is the analogous result at a positive minimum of |f|?

Since all maximum modular values |f(z)| of analytic functions are on the boundary, |f(p)| is guaranteed to be on the boundary. Function f is analytic, so it is conformal, and angles are preserved. By construction,

ξ is tangent to Γ, so arg(ξ) - arg(p) = π/2, and the direction of ξ is ip. Then, since angles are preserved, the direction of ξ in the image is if(p).

Since ξ is amplitwisted by f(p), we have $\hat{\xi}$ = ξ f'(p). This must point in the same direction as if(p).

The proof for the minimum, |f(q)| in Figure 1, is the same as for the maximum, because the minimum must also lie on the boundary.

Figure 1 shows the analytic function sin(z) + c on the unit circle Γ which has been translated so that Γ doesn't contain any zero points (roots). Figure 1 shows that $\xi_p\, f'(p)$ points in the same direction as i $f(p)$ and that ξ_q f'(q) points in the same direction of -i f(q). There has been no change in the amplitwist or the orientation of ξ_q, so how is it that the result at the positive minimum is opposite to that of the maximum? The reason apparently is that f(z) is traversing a loop outside the origin, so it never winds around the origin. Hence, the traversal of f on f(Γ) between max f(Γ) and min f(Γ) must be a half revolution or something close to it around some unspecified center of winding (?f(c)). At the minimum, f must still be perpendicular to f(Γ), but the direction of the traversal with respect to the ray f(q) reverses from that of f(p).

A circle was used for the simple loop because I haven't yet mastered plotting smooth loops in *Mathematica*. An analytic function was needed, so sin(z) was tried. Part 2 of the problem stipulates a positive minimum for |f|, so the unit circle was translated by c = 2.5 - 1.5 I so that f would have no zero points in Γ.