Needham, Chapter 7, Exercise 17, p. 374
G. Palmer, May 15, 2017, 8:15 am PST

17. Consider a spherical balloon resting on a plane. If we gradually deflate S, each point will end up on the plane so that we have a continuous mapping H of S in the plane. Observe that the north and south poles, which are antipodal, have the same image. The *Bursuk-Ulam Theorem* says that any continuous mapping H of S into the plane will map some pair of antipodal points to the same image. Consider the mapping F(p) = H(p) - H(p*), where p* is antipodal to p. The theorem amounts to showing that F has a root somewhere on S. Prove this. [Hints: It is sufficient to examine the effect of F on just the northern hemisphere. By taking the boundary of this hemisphere (the equator) to be the circle C of the previous exercise, deduce that v[H(C), 0] ≠ 0. We think v[H(C), 0] is needed to deduce v[F(C), 0] ≠ 0].

Answer 1

H(p) is a mapping of S (northern hemisphere) into the plane. F(p) is the difference between two mappings of antipodal points into the plane. “H(C)” in the hint appears to be a mapping of C to the plane, so it is a mapping of the plane to itself. But C is also the equator of the sphere. We can rotate the sphere so that any two opposite points lie on C, where we can regard them as complex numbers. After the rotation, this particular application of H to two points p and p* is an instance of the mapping of f to p and -p on the circle in exercise 16, because in this case, p* = -p. Then H(C) consists entirely of fixed points and H(-p) = -H(p). This tells us by exercise 16 that v[H(C), 0] = odd and v[H(C), 0] ≠ 0. Then F(p) = H(p) -(- H(p) = 2H(p) shows that F has the same number of roots in C as H and the number of roots is odd. We were to show that F has a root somewhere on S. That root must be H^{-1}(P), where P is a root (0) in C and H^{-1} is a mapping from the plane to the northern hemisphere. The root must be the north pole. It doesn’t matter whether H is stereographic projection or vertical projection, but in this case the metaphor of the deflating balloon suggests vertical projection.

Answer 2

H(p) is a mapping of S (northern hemisphere) into the plane. F(p) is the difference between two mappings of antipodal points into the plane. Turn the northern hemisphere upside down, so that it has the appearance of a bowl with what was the north pole now placed on ℂ at the origin. What was ℂ is now a plane passing through the maximums of the spherical modular surface. What was the equator is now the maximum points of the inverted modular surface of a function H(p) from S to ℂ. Then, by inspection, as in Ex. 13(v), we deduce that the function H has a root in S and the root is 0. The function F has the same number of roots as H.