Needham, Chapter 7, Exercise 16, p. 374
G. Palmer, May 3, 2017,6:30 pm PST

16. Let f(z) be an odd power of z, and consider its effect on the unit circle C. Note two facts: (1) if p is on C, then f(-p) points in the opposite direction to f(p); (2) ν[f(C), 0] = odd, in particular it cannot vanish. This is only one example of a general result. Show that (1) always implies (2), even if f is merely continuous. [Hints: If f is subject to (1), what can we deduce about the net rotation $\mathcal{R}$ of f(z) as z traverses the semicircle from p to -p? How is the rotation produced by the remaining semicircle related to $\mathcal{R}$?

Given:

f(z) = z^n, odd(n)
(1) f(-p) points in opposite direction to f(p); ie. f(-p) = -f(p)
(2) ν[f(c), 0] = odd

From (1) we deduce that the net rotation $\mathcal{R}$ of f(z) as z traverses the semicircle from p to -p equals π. That is:

$$\arg(f(-p)) = \arg(f(p) + \pi)$$

I'm not sure what is meant by "the remaining semicircle related to $\mathcal{R}$". Perhaps it means that after the rotations induced by the powers of z = $re^{i\theta}$ have been considered by multiplying θ by n, one must then add the semicircle rotation π to the total to account for the negative sign. The following shows why the negative sign is there:

$$f(-p) = (-p)^n = (-1)^n p^n = -f(p)$$

It is obvious that f(z) has n roots (p-points). Then by the Topological Argument Principle, which applies to continuous mappings (p. 350), ν[f(C), 0] = n, and n is odd, because the number of roots is odd. Therefore, for f = z^n, if f(-p) points in opposite direction to f(p), ν[f(C), 0] = odd.

To generalize this, we need to be able to say, if f(-p) = -f(p), then the number of roots is odd. An even number of roots would cancel to zero winds. The difference of π from f(p) to f(-p) therefore indicates a wind around one additional root plus one-half wind. Hence if f(-p) = -f(p), the number of roots counted in their multiplicities must be odd, and because ν[f(C), 0] = n for continuous mappings, the number of winds must be odd, and with the cancellations, the total winding must be ν[f(C), 0] = 1. Therefore, if f(-p) points in opposite direction to f(p), ν[f(C), 0] = 1 = odd.