Needham, Chapter 7, Exercise 14, p. 373
G. Palmer, April 26, 2017, 11:00 pm PST

Out[1321]=

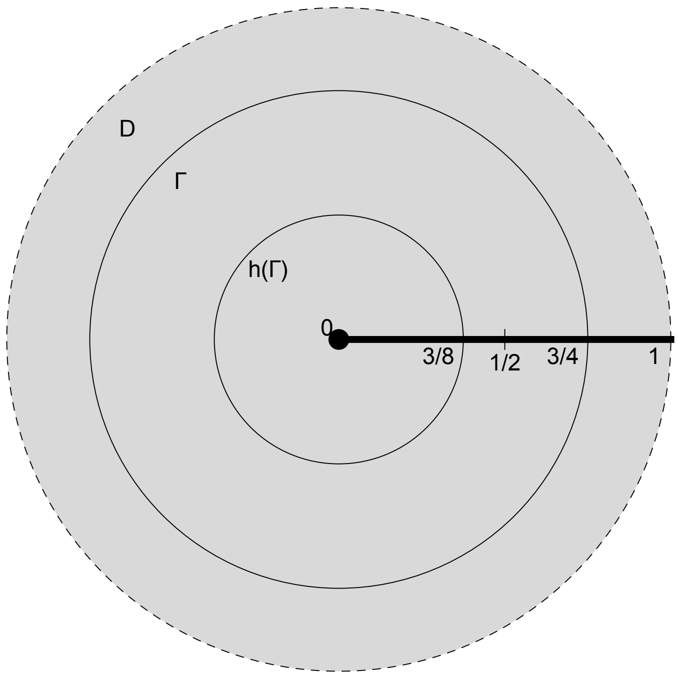

Figure 1

14 With $r = |z|$, the mapping $h(z) = \phi(r)\, z$ will be a continuous function of z if $\phi(r)$ is a continuous function of r. Consider the continuous mapping h of the unit disc to itself corresponding to

$$\phi(r) = 0, \qquad 0 \le r < (1/2);$$
$$\phi(r) = 2r - 1, \qquad (1/2) \le r \le 1.$$

(i) Describe this mapping in visually vivid terms.

Put r on the real axis between and including 0 and 1. On the first half of the domain, h(z) is zero, so h(z) is a line of length r on the real axis between 0 and (1/2). Figure 1 shows h(z) as a black point at the origin and r as a black line on [0..1/2). On the second half of the domain, there is amplification of every z from 0 to z (or a contraction of every z down to zero along a ray). Points h(Γ) are plotted at all radii from 0 to 1, coloring the disc a solid lightgray, and r is plotted as a gray line on [1/2..1]. So on the lower domain, the image is a point; on the upper domain, the image is a solid disc.

(ii) Taking Γ to be the circle $|z| = (3/4)$ and letting $p = 0$, try (and fail) to make sense of (11).

$$\nu[h(\Gamma), p] = \sum_{\text{inside } \Gamma} \nu\left(a_j\right) \qquad (11)$$

If we consider the mapping of h(Γ) for r = (3/4) and p = 0,

$$h(z) = \phi(r)z = (2\,(3/4) - 1)\, z = z/2$$

r is fixed and $z/2$ has one root, 0, inside Γ, so $\nu[h(\Gamma), p] = +1$. The mapping appears to be analytic when $r = |z|$ and $|z| = [(1/2)..1]$. Why does this not make sense? $h(z)$ has a single root for every value of r in the domain $(1/2) \le r \le 1$, but in the domain $0 \le r < (1/2)$, where $h(z) = 0$ there is no defined root, so $\nu[h(\Gamma), p] = 0$ or undefined. One could say it doesn't make sense, or one could say that (11) does not apply on the lower half of the composite domain, so it doesn't apply to the function taken as a whole.

Figure 1 was plotted first using the actual functions and plotting points, but the .pdf file proved to be too large to upload, so the image was replicated using graphics functions for circles and lines. The file was reduced from 2.5 MB to 126 KB.