Needham, Chapter 7, Exercise 13, p. 373
G. Palmer, April 29, 2017, 5:15 pm PST

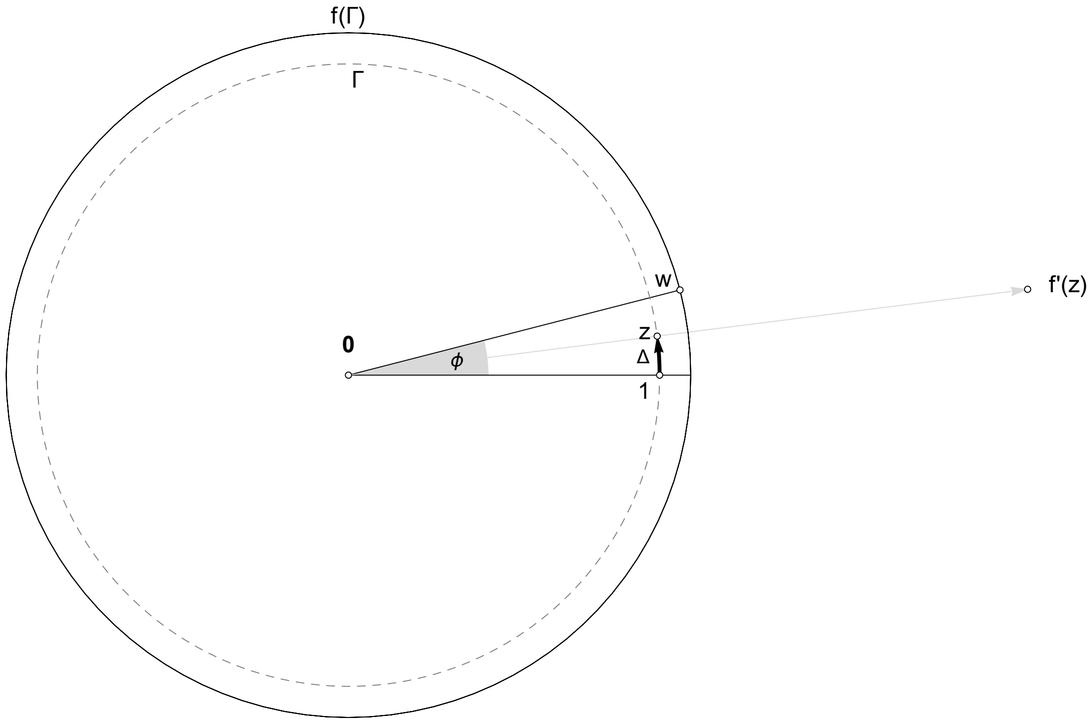

Figure 1: Infinitesimal traversal and rotation

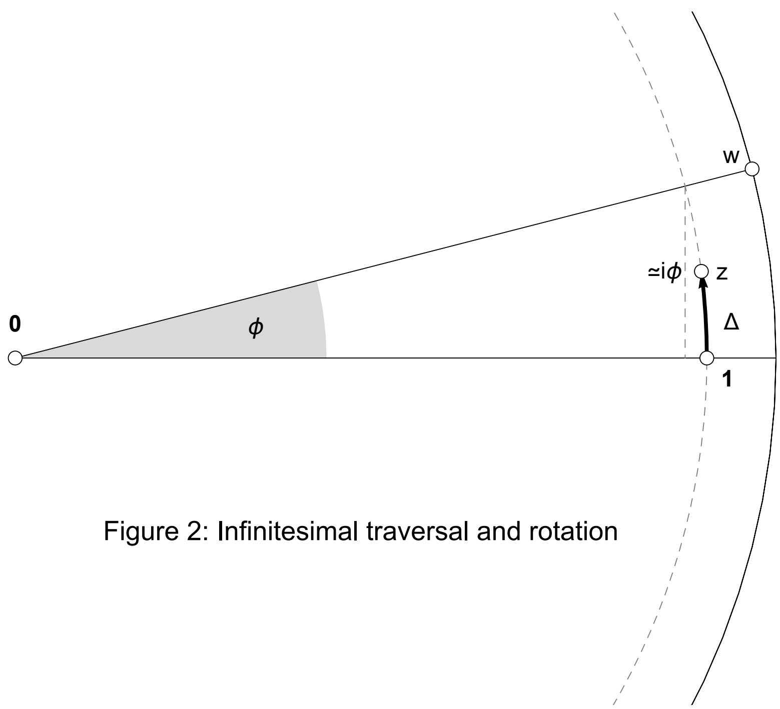

Figure 2: Infinitesimal traversal and rotation

13 Let w = f(z) be analytic inside and on a simple loop Γ, and suppose that f(Γ) is an origin-centred circle.

(i) If Δ is an infinitesimal movement of z along Γ and ϕ is the correspondingly infinitesimal rotation of w, show geometrically that

$$\frac{f'\Delta}{f} = i\phi$$

Let Γ be a unit circle, which at infinitesimal scale will serve as a surrogate for a general simple loop. Then $\frac{|f'(z)|}{|f(z)|}$ is the ratio by which Δ is amplified or contracted (Figure 1). The infinitesimal angle ϕ equals the distance that ϕ subtends on Γ. As ϕ tends to 0, iϕ approximates the path of z along Γ under the rotation $\mathcal{R}^{\phi}$. Δ is described as a movement along Γ, so it must be a complex number. As Δ tends to 0, the more closely its path approximates the vectors iy or i|Δ| (Figure 2).

From the plot and our understanding of amplitwist, we can see that

$$\frac{|f'|}{|f|} \simeq \frac{i\phi}{\Delta} \Longrightarrow \frac{|f'|}{|f|}\Delta \simeq i\,\phi,$$

but this does not necessarily mean that $\frac{f'}{f}\Delta \simeq i\,\phi$. This would only be the case if arg(f') = arg(f). In fact, with power functions, such as f = z^n, the quotient behaves as though this is the case. If arg(f) = nθ, then arg(f') = (n-1)θ and f'/f has arg θ, which means that as θ tends to 0, the twist goes to 1 and the amplitwist goes to A = $\frac{|f'|}{|f|}$. Therefore, we can say that for these particular analytic functions $\frac{f'}{f}\Delta \simeq i\,\phi$. In (iii) we see that analytic functions have one more wind than their derivative, indicating again that the degree of f is degree f' + 1, so the equation should hold for any analytic functions.

(ii) As z traverses Γ, explain why ν[Δ, 0] = 1 and ν[iϕ, 0] = 0.

ν[Δ, 0] = 1 because it is defined as infinitesimal movement of z along a simple loop Γ. Δ completes a full traversal of Γ when it has covered the length of Γ and Arg(Δ) = 2 π. ν[iϕ, 0] = 0 because iϕ is a vector with arg π/2, not a traversal of Γ.

(iii) Referring to (i) of the previous exercise, show that ν[f(Γ), 0] = ν[f'(Γ), 0] + 1.

From Ex. 12 (i), $\nu[\tilde{\Gamma}, 0] = \nu[p(\Gamma), 0] + \nu[q(\Gamma), 0]$, where $\tilde{\Gamma}$ is the image curve under z ↦ p(z) q(z).

From the first sentence of Ex. 13, w = f(z) is analytic inside and on Γ. Then f(Δ) = f'(z) Δ, so we can write:

$$\nu[f(\Gamma), 0] = \nu[f'\Delta(\Gamma), 0] = \nu[f'(\Gamma), 0] + \nu[\Delta, 0],$$

so it follows from (ii) that

$$\nu[f(\Gamma), 0] = \nu[f'(\Gamma), 0] + 1$$

(iv) Deduce from the Argument Principle that f has one more root inside Γ than f' has....

The AP states that the total number of p-points with their multiplicities equals $\nu[h(\Gamma), p]$, the winding number of h on Γ. The winding number $\nu[\Delta, 0] = 1$ indicates one additional p-point, and $\nu[f'(\Gamma), 0]$ gives the number of p-points of f', so f has one more root than f' inside Γ.

(v) From this we deduce, in particular, that f has at least one root inside Γ. Derive this fact directly by considering the portion of the modular surface lying above Γ and its interior.

The modular surface is defined as $Z = |f(z)|$ at z (p. 56). If z is confined to Γ and the interior of Γ, the entire modular surface lies over Γ and its interior. From Section VI.1 Maximum-Modulus Theorem, we know that

There can be no pits in the surface [local minima of $|f|$ *except at 0-points*. [with the exception of $z \mapsto$ const.] (p. 357)

Then it follows that if we can see a pit in the modular surface of an analytic function, then it must be a 0-point. From (12) we know that all the preimages of an analytic $f(\Gamma)$ lie inside Γ. Therefore the preimage of a 0-point must lie inside Γ, and it must be the point $z = 0$, because $|f(z)| = |x + iy| = 0$. Since we can always translate a simple loop Γ to include the origin, there will always be at least one root of f inside Γ.