Needham, Chapter 7, Exercise 12, p. 372.
G. Palmer, October 23, 2017, 4:30 pm PST

*12. We can formalize, and slightly generalize, our explanation of Rouche's Theorem as follows:*

*(i) If p(z) and q(z) are nonzero on a simple curve Γ, and $\tilde{\Gamma}$ is the image curve under $z \mapsto p(z)\, q(z)$, show that*

$$\nu[\tilde{\Gamma},0] = \nu[p(\Gamma), 0] + \nu[q(\Gamma), 0].$$

To restate the equation, the total number of winds $\nu[\tilde{\Gamma},0]$ under this construction is the sum of the winds contributed by p and q for each p-point in Γ as z traverses Γ. See [8]. The winds for each are $\nu[p(\Gamma), 0]$ and $\nu[q(\Gamma), 0]$. Let m and n be the number of p-points of p(z) and q(z) in Γ. Then $\nu[p(\Gamma), 0] = m$ and $\nu[q(\Gamma), 0] = n$ by the Topological Argument Principle (10), p. 350. The winding numbers are associated with the angles $m2\pi$ and $n2\pi$. Adding these angles produces $\arg(p(\Gamma)q(\Gamma)) = (m+n)2\pi$, which is the angle of $\tilde{\Gamma}$ under $z \mapsto p(z)\, q(z)$ for one traversal of Γ. Since adding the angles amounts to adding the winding numbers, the angle $(m+n)2\pi$ is associated with the winding number $\nu[\tilde{\Gamma},0] = \nu[p(\Gamma), 0] + \nu[q(\Gamma), 0]$.

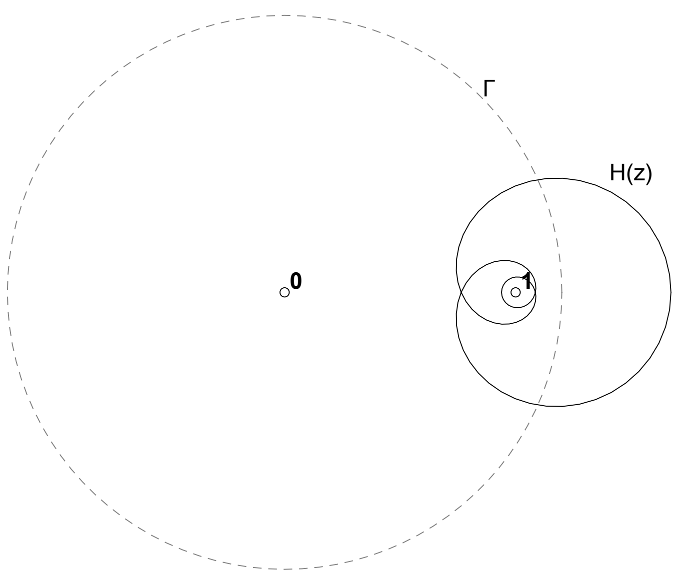


Figure 1:H(z) = 1 + f(z)/g(z)

*(ii) Write*

$$f(z) + g(z) = f(z)[1 + \frac{g(z)}{f(z)}] = f(z)\, H(z)$$

*If $|g(z)| < |f(z)|$ on Γ, sketch a typical H(Γ). Deduce that $\nu[H(\Gamma), 0] = 0$.*

A typical H(Γ) would be a loop or loops around 1, as in Figure 1, in which $g(z) = e^{z}$ and $h(z) = z^3\, e^a$, a = 1.05. Because $|g(z)| < |f(z)|$, $\left|\frac{g(z)}{f(z)}\right| < 1$. Hence the little dog on the H(z) leash attached to owner f(z) can not reach the tree at the origin. Because she can not reach the origin, H(z) does not wind around 0, so $\nu[H(\Gamma), 0] = 0$.

*Using the previous part, deduce Rouche's Theorem.*

$$\nu[f(\Gamma)\,H(\Gamma),0] = \nu[f(\Gamma), 0] + \nu[H(\Gamma), 0] \quad \textit{from (i) and (ii)}$$

$$= \nu[f(\Gamma), 0]$$

Since $|g(z)| < |f(z)|$ and $f(z) + g(z)$ can always be rewritten as $f(z)[1 + \frac{g(z)}{f(z)}]$, this proves that f + g has the same number of roots inside Γ as f.

*(iii) If we only stipulate that $|g(z)| \leq |f(z)|$ on Γ, then parts of H(Γ) could actually coincide with the circle $|z-1| = 1$, rather than lying strictly inside it, and $\nu[H(\Gamma), 0]$ might not be well-defined. However, show that if we further stipulate $f+g \neq 0$ on Γ, then $\nu[H(\Gamma), 0] = 0$ as before.*

We need only examine the case where $|g(z)| = |f(z)|$ and $f+g \neq 0$ for all z on Γ. It appears that these conditions can only be met throughout f(Γ) if arg(f) - arg(g) = const. $\neq \pi$. Let $f = re^{i\theta}$ and let $g = re^{i(\theta-\alpha)}$, $\alpha$ a real constant. Then $g/f = e^{i\alpha}$. Since we have stipulated that $f+g \neq 0$, $g \neq -f$. So as long as f and g never become diametrically opposite ($\alpha = \pi$), things are OK (Vasco, VCA Forum). Hence, f and g must travel at the same speed and in the same direction, for otherwise, they would eventually intersect, though perhaps not in a single traversal of Γ by z. Under these conditions, H(Γ) will never touch or encircle 0 even though $|g(z)| = |f(z)|$. It is only when $g(z) = -f(z)$ that H(Γ) touches 0, rendering it undefined.

Deduce that $\nu[(f+g)(\Gamma), 0] = \nu[f(\Gamma), 0]$.

In (ii), we found that

$$\nu[f(\Gamma)\,H(\Gamma), 0] = \nu[f(\Gamma), 0] + \nu[H(\Gamma), 0] = v[(f+g)(\Gamma), 0]$$

Since $\nu[H(\Gamma), 0] = 0$, follows that $v[(f+g)(\Gamma), 0] = \nu[f(\Gamma), 0]$. In the example given $v[(f+g)(\Gamma), 0] = 1$.