Needham, Chapter 7, Exercise 11, p. 372.
G. Palmer, April 20, 2017, 1:30 pm PST

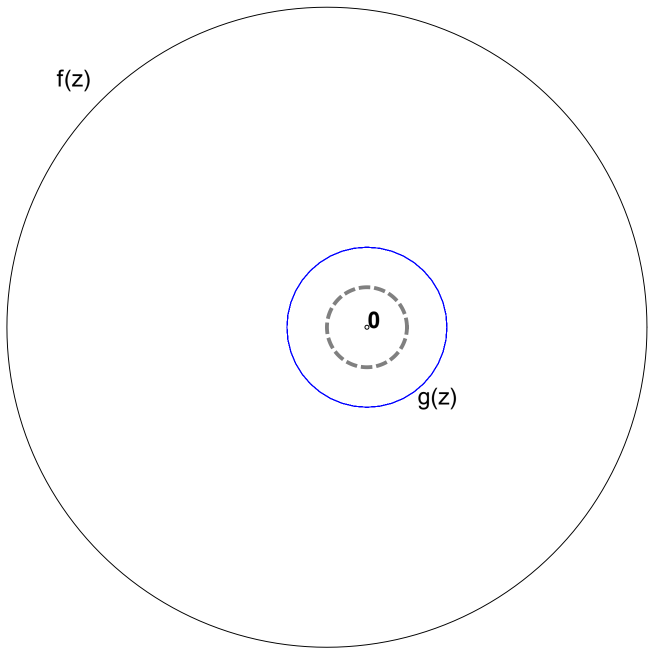

Figure 1:
Rouche's theorem: $2z^5 + 8z - 1 = 0$
$f(z) = (8z-1)$, $g(z) = 2z^5$
on unit circle

11 (i) Applying Rouche's Theorem to $f(z) = 2z^5$ and $g(z) = 8z-1$, show that all five solutions of the equation $2z^5 + 8z - 1 = 0$ lie in the disc $|z| < 2$.

Find $|8z - 1|$, when $|z| = 2$ (i.e. $x^2 + y^2 = 2$, $y = \sqrt{x^2 - 2}$)

When z = -2 or 2, $|g(z)| = |8z - 1| = |-16 - 1| = 17$ or $|16 - 1| = 15$.

Find $|f(z)| = \left| 2z^5 \right|$ when $x = \pm 2$

$$|f(z)| = 64$$

$|g(z)| = 17 < 64 = |f(z)|$ when $|z| = 2$ and f(z) has five zeroes, so (f+g) must have five zeros in the disc.

(ii) By reversing the roles of f and g, show that there is only one root in the unit disc. Deduce that there are four roots in the ring $1 < |z| < 2$.

Let $f = 8z - 1$ and let $g(z) = 2z^5$ on the unit disc. Then we obtain minimum and maximum values for $|f(z)|$ be letting z = 1 and -1.

$$|f(z)| = \{7, 9\}$$

$|g(z)| = 2$

$|g(z)| = 2 < 7 = |f(z)|$

Since $|g(z)| < |f(z)|$, then the Argument principle says that (f+g) must have the same number of roots as f(g). f(g) has only one root, so (f+g) has only one root inside $|z| = 1$. In (i) we showed there are five roots inside $|z| = 2$, and we know there is one inside $|z| = 1$, so there must be four roots in the ring $1 < |z| < 2$.