Needham, Ch. 6, Section III, Subsection 12, The Hemisphere Model and Hyperbolic Space, p. 325, ℙ -1, ln 7, [exercise]
November 26, 2016, 3:00 pm, PST

Out[6341]=

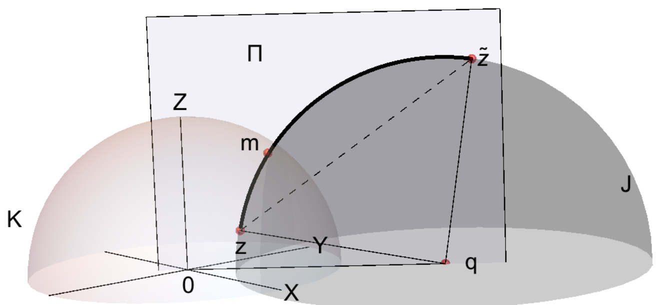

Figure 1. Reflection of hyperbolic space in h−plane K

Out[6344]=

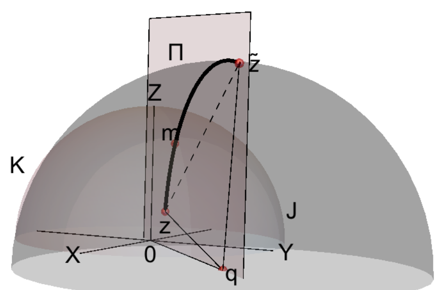

Figure 2. Reflection of hyperbolic space in h−plane K

We therefore suspect that $\mathcal{I}_K$ is reflection in this h-plane. This can be confirmed [exercise] by generalizing the argument in [23b].

The h-plane in question is K, a hemisphere centered at q = 0. Let z be a point in h-space. Let $\tilde{z} = \mathcal{I}_K(z)$. Then any sphere, say J, with equator on the plane ℂ, the plane of the horizon, and containing z and $\tilde{\tilde{z}}$ in its surface forms an h-circle through z and $\tilde{z}$ in a plane Π orthogonal to ℂ. The h-circle in J is the unique h-line through z and $\tilde{z}$ in the plane through z and $\tilde{z}$ orthogonal to ℂ. $\mathcal{I}_K(z)$ maps the orthogonal hemisphere J to J, swapping zm with $\hat{\tilde{z}}$m. Since $\mathcal{I}_K(z)$ is a motion in h-space, the segment zm equals z$\tilde{m}$. Therefore, inversion in the hemisphere K with equator on the horizon is reflection $\mathcal{R}_K$ of hyperbolic space at the h-plane K.

Very simply, inversion in a hemisphere (h-plane) K with equator on the horizon can be shown to induce a movement in an orthogonal hemisphere J with equator on the horizon and containing the points of inversion z and $\tilde{z}$. It does not appear to matter whether the center of J falls within the orthogonal plane containing the Z axis and the points of inversion, just so long as z and $\tilde{z}$ are in J.

The figures

J is plotted with center in the plane Π and equator on ℂ. To calculate q and r of J, use the cosine rule on a complex (w) plane superimposed onto the plane Π defined as above. w and $\tilde{w}$ in the w-plane correspond to z and $\tilde{z}$ in Π. After the calculations, rotate the w-plane to the θ angle of Π. Complex points must be converted to points in 3D.

$$r^2 = |w|^2 + q^2 - 2\,|w|\,q\cos(\arg(w)) \qquad (1)$$

$$r^2 = |\tilde{w}|^2 + q^2 - 2\,|\tilde{w}|\,q\cos(\arg(\tilde{w})) \qquad (2)$$

Subtract (2) from (1) and solve for q. Then r = |w - q| = $|\tilde{w} - q|$.