Some questions and notes re. Chapter 6
G. Palmer
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(1) Section I.4, footnote 3, p. 273, bottom, "Can the one on the far right be obtained by bending a rectangle."

My answer: I now believe that the right hand diagram in [4] is intended to represent an elliptic cylinder, so it could be obtained by bending a rectangle. I was originally under the impression that the diagram represented a cone shape. I wrote that bending a rectangle would require stretching the panel into what appears to be an isosceles trapezoid. Since the panel is "flat as a pancake", it could not be obtained by bending a rectangle, as that would alter angles, lengths of sides, and the value of E.

(2) Question, Section I.5, p. 274, para. 3, ln. 5: "as Π rotates about n."

Stupid question resulting from careless reading removed.

(3) Section I.5, p. 275, line 1: "visually convince yourself that κ = 0 everywhere on each of the intrinsically flat surfaces in [4]."

Vasco pointed out that "at least one of $k_{\min}$ and $k_{\max}$ must be zero at every point on the surface and so k(p) ≡ $k_{\min}\, k_{\max}$ ≡ 0".

With intrinsically flat surfaces, there is no stretching, there is no change in lengths, angles, or areas, so no change in curvature and E(T) = 0 for all three triangles. Since [4] shows only flat (Euclidean) surfaces,

$$\kappa = E(T)/A(T) = 0/A(T) = 0$$

(4) Section II Spherical Geometry, Subsection 3, p. 283, ln -1 to p. 284 ln 1.

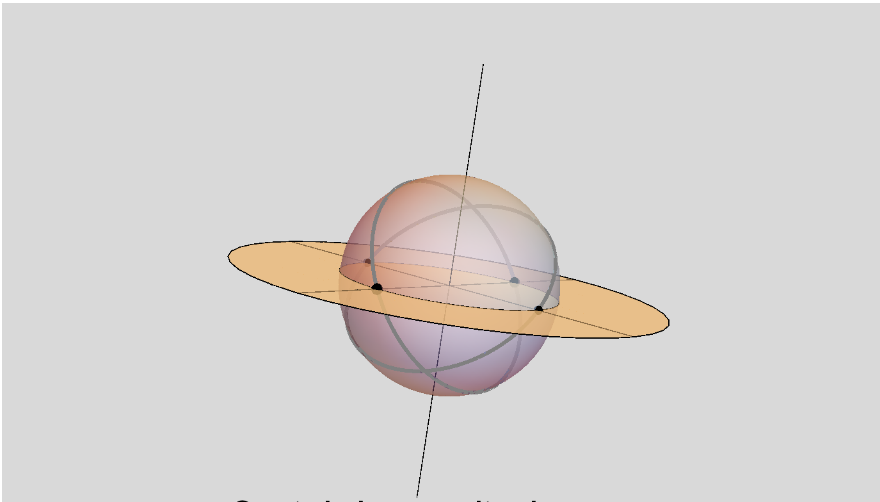
"Consequently, we may also identify the 'lines' of the geometry as the shortest routes from one point to another, and we can likewise [exercise] determine angles."

Normally we determine angles at the intersections of “lines” (curves) on surfaces as the angle between tangent lines in the tangent plane at the point of intersection: $\theta = \arg(L_2) - \arg(L_1)$. This requires having a projection from the surface line to the line on the tangent plane. For the Riemann sphere Σ, an easier method would be to project the lines on Σ to the complex plane. Since stereographic projection is conformal, angles of intersection in Σ can be read off the projections on $\mathbb{C}$.

(5) Section II.3, paragraph 4, line 2 [exercise] "It follows from (9) that it is impossible [exercise] to draw a map of the sphere that faithfully represents every aspect of its intrinsic geometry."

$$E(T) = \kappa\, A(t), \text{ where } \kappa = \left(1\big/R^2\right) \qquad (9)$$

The sphere has positive external curvature and positive error E(T). Projecting features onto a plane or an object of different intrinsic curvature requires bending and stretching the sphere, so one can not simultaneously preserve lines, circles, and angles.



Great circles on unit sphere intersecting ℂ on unit circle at opposite points

(6) Section II.3, p. 285, paragraph 2, ln 5: “In fact it’s not too hard to see [exercise] that great circles on Σ are mapped to circles in ℂ that intersect the unit circle at opposite points.”

All great circles on a sphere have the same center and radius, so if a great circle passes through p on ℂ, it must also pass through -p.

(7) Section II.3, p. 286, line 1 [exercise]

From (6) on p. 126,

$$[\tilde{a}\tilde{b}] = \frac{R^2}{[qa][qb]}[ab]$$

where [ab] = |a-b| and R is the radius of a circle of inversion K centered at q. In the present case (Figure [13b]), q = i, R = $\sqrt{2}$, d$\hat{s}$ corresponds to [$\tilde{a}\tilde{b}$], ds corresponds to [ab], [qa] corresponds to [Nz], and [qb] corresponds to [N(z+ds)]. As ds approaches 0, [N(z+ds)] approaches [Nz], so we can write $[Nz]^2$ in the denominator. Making all these substitutions, we get

$$d\hat{s} = \frac{2}{[Nz]^2}ds$$

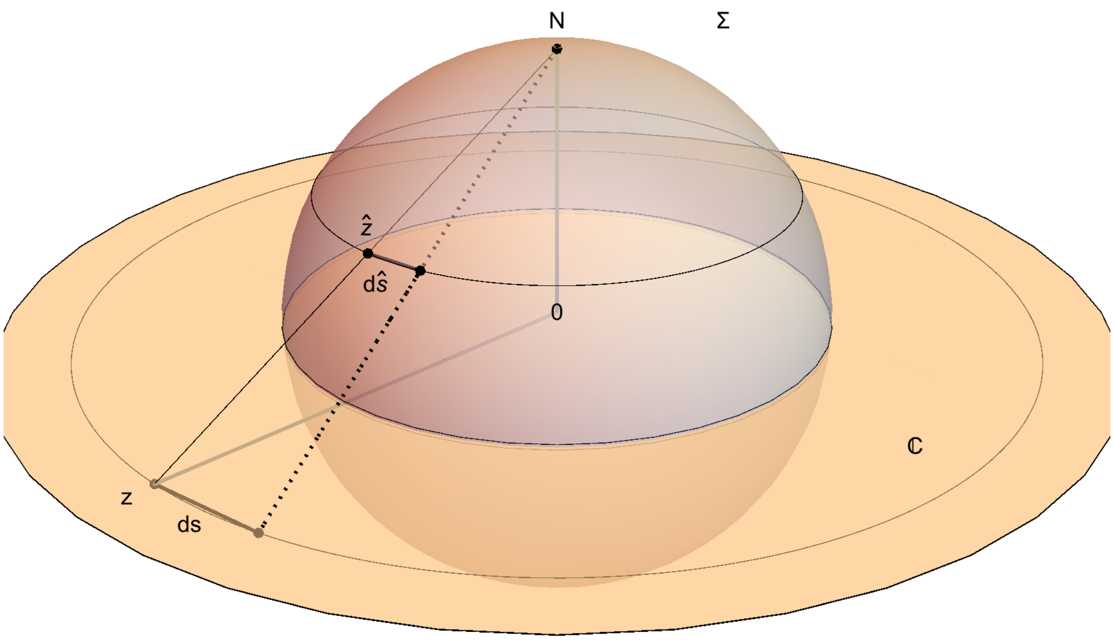


Figure [13a] with $d\hat{s}$ parallel to $\mathbb{C}$

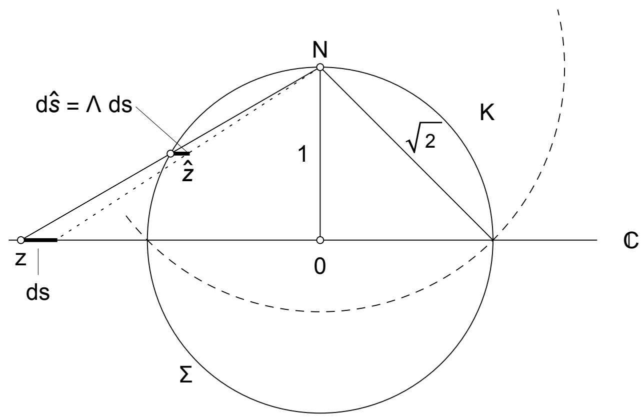


Figure [13b] with $d\hat{s}$ parallel to $\mathbb{C}$

(8) Section II.3, line 2, p. 286 [exercise] using Pythagora's Theorem with $d\hat{s}$ parallel to $\mathbb{C}$.

In Chapter 3, Section II Inversion, p. 124, Figure [1b], we see that the distance from the center q of the circle of inversion K to an interior point is ρ and the distance to its reflection is R^2/ρ. In Figure [13b], p. 286, we see that R = $\sqrt{2}$.

When $d\hat{s}$ is parallel to $\mathbb{C}$, it is a secant of an arc of a circle ([13a] modified, above). Then ds must also be a parallel secant of an arc of a congruent circle in $\mathbb{C}$ (In my version of [13b] one has to imagine that ds and $d\hat{s}$ point out of the x-z plane from points on the circle to points on the circle in the case of secants.) Draw a triangle Nz(z+ds). A similar triangle $N\hat{z}(\hat{z}+d\hat{s})$ is formed at the same time. The points (z+ds) and $(\hat{z}+d\hat{s})$ are also reflections in a sphere of inversion with radius $\sqrt{2}$.

From p. 124, applied to Figure [13b], p. 285,

$$[Nz] = \frac{R^2}{\rho}$$

$$\rho = \frac{R^2}{[Nz]}$$

$$\frac{d\hat{s}}{\rho} = \frac{ds}{[Nz]} \text{ by similar triangles}$$

$$d\hat{s} = \frac{\rho}{[Nz]}ds = \frac{R^2/[Nz]}{[Nz]}ds = \frac{2}{[Nz]^2}ds$$

This proves the equation without using (6). By the Pythagorean theorem, the distance $[Nz]^2 = 1 + | z |^2$. We can write the equation

$$d\hat{s} = \frac{2}{[Nz]^2} ds = \frac{2}{(1+|z|^2)} ds$$

(9) p. 286, Section 4, Spatial Rotations as Möbius Transformations
I had some difficulty understanding the first two paragraphs of this section. Part of it was what I call “notation distraction”. I was reading the wrong interpretation into the notation and not paying sufficient attention to the text, which tells us that the objective is to use stereographic projection to find a function in $\mathbb{C}$. The key is “Purely in terms of $\mathbb{C}$.” Both ds and $d\tilde{s}$ lie in $\mathbb{C}$. Then we suppose there is an analytic function $f(z) = \tilde{z}$ that is a motion, so it must satisfy the first equation obtained from the projection.

We can rewrite the first equation ($\Lambda(\tilde{z})d\,\tilde{s} = \Lambda(z)ds$) as

$$\frac{d\,\tilde{s}}{ds} = \frac{\Lambda(z)}{\Lambda(\tilde{z})}$$

From p. 284, we have ds = |dz| Then, remembering that $\tilde{z} = f(z)$, we can rewrite the second equation ($d\,\tilde{z} = f'(z)dz$) as

$$|f'(z)| = \frac{|d\,\tilde{z}|}{|dz|} = \frac{d\,\tilde{s}}{ds} = \frac{\Lambda(z)}{\Lambda(\tilde{z})} = \frac{\Lambda(z)}{\Lambda[f(z)]}, \text{ which for S = } \Sigma \text{ and using (16), is } \frac{1+|f(z)|^2}{1+|z|^2}.$$

Vasco noticed the need for writing |f’(z)| in the forum errata for chapter 6.

We have established a stereographic projection of $\mathbb{C}$ to Σ and Σ to $\mathbb{C}$ by using inversion in K, which is a second sphere centered at N and intersecting Σ at $\mathbb{C}$. The stereographic projection is used to derive a function f’(z) in $\mathbb{C}$. Note that $\tilde{z} = f(z)$ and z also lie in $\mathbb{C}$.

(11) p. 286, line -1 to p. 287, line 1: "Consider [14a], which shows a new intrinsic method of constructing the reflection $\mathcal{R}_{\hat{L}}(\hat{z})$ of a point $\hat{z}$ on Σ, namely [exercise], as the second intersection point of any two circles centred on $\hat{L}$ and passing through $\hat{z}$."

On p. 129, refer to Figure [6] and the lines in italics:

The inverse $\hat{z}$ of z in K is the second intersection point of any two circles that pass through z and are orthogonal to K.

This applies only to reflection in a plane, but we can write something similar for any two circles that pass through z and are orthogonal to a great circle in Σ. In Figure [14a], p. 287, $\hat{L}$ bisects both circles. A line on the surface from $\hat{z}$ to the point shown as $\mathcal{R}_{\hat{L}}(\hat{z})$ in the figure is orthogonal to $\hat{L}$ and bisected by it. [This seems obvious, but perhaps it needs to be proven.] Therefore, $\mathcal{R}_{\hat{L}}(\hat{z})$ can be constructed as the second intersection point of any two intersecting circles centred on $\hat{L}$ and passing through $\hat{z}$.

(12) p. 288 matrix calculation for
(19)

```
ξPlus = a;
ξMinus = -1 / ā;
√m == e^(-iPi/2);
```

$$\text{matrixM} = \bar{a}\begin{pmatrix} -\xi\text{Minus} & \xi\text{Plus} \\ -1 & 1 \end{pmatrix}.\begin{pmatrix} e^{-\text{iPi}/2} & 0 \\ 0 & 1\big/e^{-\text{iPi}/2} \end{pmatrix}.\begin{pmatrix} 1 & -\xi\text{Plus} \\ 1 & -\xi\text{Minus} \end{pmatrix}$$

$$\text{matrixM} = \begin{pmatrix} e^{\text{i Pi}/2}\,|\,a\,|^2 + e^{-\text{i Pi}/2} & 2\,\text{i a Sin[Pi / 2]} \\ 2\,\text{i}\,\bar{a}\,\text{Sin[Pi / 2]} & e^{-\text{i Pi}/2}\,|\,a\,|^2 + e^{\text{i Pi}/2} \end{pmatrix}$$

(13) pp. 288, bottom and 289 line 1 “[exercise] ...the most general opposite motion is represented by a function of the form

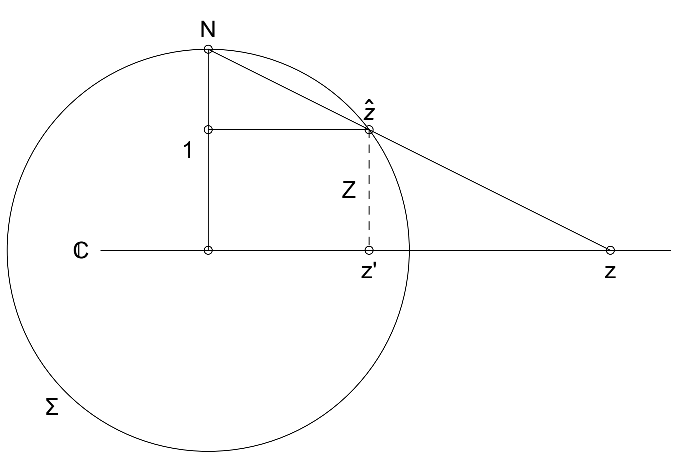
$$z \mapsto \left(\frac{A\bar{z}+B}{-\bar{B}\bar{z}+\bar{A}}\right)$$

From (13), p. 282, we know that “Every direct motion of the sphere is a rotation, and every opposite motion is the composition of a rotation and a reflection.” Since the order of composition does not matter, we can represent the opposite motion as $\tilde{M} = M \circ \mathcal{R}_L$, where M is a direct motion.

$$M = \frac{Az+B}{-\bar{B}z+\bar{A}} \quad \text{(20), p. 288}$$

$\mathcal{R}_L = \bar{z}$ (a reflection of z in $\mathbb{C}$ induced by a reflection of Σ in the xz plane)

$$M \circ \mathcal{R}_L \equiv M(\bar{z}) = \frac{A\bar{z}+B}{-\bar{B}\bar{z}+\bar{A}}$$



(14) p. 289. To see that $a = \frac{1+im}{1-n}$ and $|\,a\,|^2 = \frac{1+n}{1-n}$, we refer back to (19) on p. 146 and make a table of correspondences.

Table 1: Correspondences of coordinates

Cartesian coordinates	X	Y	Z
unit vector components	l	m	n
complex coordinates	x	y	

We are given:

z = x + i y
z' = X + iY

$$z = \frac{|z|}{|z'|} z'$$

$$\frac{|z|}{|z'|} = \frac{1}{1-Z} \quad \text{(by similar right triangles)}$$

$$z = \frac{|z|}{|z'|} z' = \frac{1}{1-Z}(X + iY) = \frac{X+iY}{1-Z}$$

$$|z|^2 = \frac{1+Z}{1-Z}$$

Now we set a = z and substitute from Table 1.

$$a = z = \frac{X+iY}{1-Z} \equiv \frac{l+im}{1-n}$$

$$|a|^2 = |z|^2 = \frac{1+Z}{1-Z} = \frac{1+n}{1-n}$$

p. 289, bottom: *Substituting these expressions into (19) and removing the common factor of 2/(1-n), we obtain [exercise]*. This is straightforward, so I'm not going to bother with it here.

p. 290, line 1: *You may check for yourself that this matrix is "normalized": det[*$R_{\mathbf{v}}^{\psi}$*] = 1.*

det[$R_{\mathbf{v}}^{\psi}$] = ad-bc = (cos(ψ/2) + in sin(ψ/2))(cos(ψ/2) - in sin(ψ/2)) - $(m + \text{il})(-m + \text{il})\sin^2(\psi/2)$

$$= \cos^2(\psi/2) + n^2 \sin^2(\psi/2) - (-m^2 - l^2)\sin^2(\psi/2)$$

$$= \cos^2(\psi/2) + (l^2 + m^2 + n^2)\sin^2(\psi/2)$$

$$= \cos^2(\psi/2) + \sin^2(\psi/2) = 1$$

p. 290, find matrix for rotation of (Pi/2) about **i**. Note that **i** = (1, 0, 0), not i on the complex plain. For **i** = (1, 0, 0), l = 1, m = 0, n = 0. Substitute into (21).

$$[R_{\mathbf{i}}^{Pi/2}] = \begin{pmatrix} \cos(\pi/4) + i\,0\sin(\pi/4) & (-0+i)\sin(\pi/4) \\ (0+i)\sin(\pi/4) & \cos(\pi/4) - i\,0\sin(\pi/4) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\pi/4) & i\sin(\pi/4) \\ i\sin(\pi/4) & \cos(\pi/4) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

(15) *Comparing this with (21), we see [exercise] that this rotation of $\psi = (2\pi/3)$ about the axis $\mathbf{v} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$.*

Write equations to find l, m, n, and ψ. We will need four equations. We equate the corresponding elements of matrix (21) and matrix (22).

E1: $\cos(\psi/2) + in\sin(\psi/2) = (1/2)(1 - i)$
E2: $(-m + il)\sin(\psi/2) = (1/2)(-1 + i)$
E3: $(m + il)\sin(\psi/2) = (1/2)(1 + i)$
E4: $\cos(\psi/2) - in\sin(\psi/2) = (1/2)(1 + i)$

E1 + E4
$\cos(\psi/2) + in\sin(\psi/2) = (1/2)(1 - i)$
$\cos(\psi/2) - in\sin(\psi/2) = (1/2)(1 + i)$
$2\cos(\psi/2) = 1$
$\psi = 2\arccos(1/2) = 2\pi/3$. This confirms the rotation.

Substitute ψ into E1:

$$\cos(\pi/3) + in\sin(\pi/3) = (1/2)(1 - i)$$

$$1/2 + i\,n\,\frac{\sqrt{3}}{2} = 1/2 - i/2$$

$$i\,n\,\frac{\sqrt{3}}{2} = -i/2$$

$$n\sqrt{3} = -1$$

$$n = \frac{-1}{\sqrt{3}}$$

Substitute ψ into E2 and E3 and add the equations.

$$(-m + il)\frac{\sqrt{3}}{2} = (1/2)(-1 + i)$$
$$(m + il)\frac{\sqrt{3}}{2} = (1/2)(1 + i)$$

$$il\sqrt{3} = i$$

$$l = \frac{1}{\sqrt{3}}$$

Then substitute l into E3.

$$(m + i/\sqrt{3})\ \frac{\sqrt{3}}{2} = (1/2)(1 + i)$$

$$m\frac{\sqrt{3}}{2} + i/2 = 1/2 + i/2$$

$$m = (1/2)(2/\sqrt{3}) = \frac{1}{\sqrt{3}}$$

$$\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$$

(16) Subsection 4, p. 291: *If we multiply $\mathbb{V}$ by another quaternion, $\mathbb{W} = w + \boldsymbol{W}$, then (24) and (25) imply [exercise] that*

$$\mathbb{V}\mathbb{W} = (vw - \boldsymbol{V} \bullet \boldsymbol{W}) + (v\,\boldsymbol{W} + w\,\boldsymbol{V} + \boldsymbol{V} \times \boldsymbol{W}).$$

We use *Mathematica* to expand the product $\mathbb{V}\mathbb{W}$ distributively over the sums.

```
𝕍 = v + v1 i + v₂ j + v₃ k;
𝕎 = w + w₁ i + w₂ j + w₃ k;

𝕍𝕎 = 𝕍 𝕎 // Expand
```

$$v\,w + i\,v1\,w + j\,v_2\,w + k\,v_3\,w + i\,v\,w_1 + i^2\,v1\,w_1 + i\,j\,v_2\,w_1 + i\,k\,v_3\,w_1 + j\,v\,w_2 + i\,j\,v1\,w_2 + j^2\,v_2\,w_2 + j\,k\,v_3\,w_2 + k\,v\,w_3 + i\,k\,v1\,w_3 + j\,k\,v_2\,w_3 + k^2\,v_3\,w_3$$

We rearrange.

```
𝕍𝕎 == v w + (i² v1 w₁ + j² v₂ w₂ + k² v₃ w₃) + (i v w₁ + j v w₂ + k v w₃) +
   (i v1 w + j v₂ w + k v₃ w) + (i j v₂ w₁ + i k v₃ w₁ + i j v1 w₂ + j k v₃ w₂ + i k v1 w₃ + j k v₂ w₃);
```

It is easy to see V • W, vw, vW and wV in the equation for $\mathbb{V}\mathbb{W}$. The last group should be the cross product.

```
𝕍𝕎 ==
  (vw - V • W) + v W + w V + (i j v₂ w₁ + i k v₃ w₁ + i j v1 w₂ + j k v₃ w₂ + i k v1 w₃ + j k v₂ w₃);
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But here it appears that *Mathematica* is not on the same page. The $v_3\,w_1$ coefficicent should be affixed to KI, not IK and the $v_3\,w_2$ coefficient should be affixed to KJ, not JK. We do the calculation ourselves, getting

$$\mathbf{V} \times \mathbf{W} = v_1\,w_2\,\mathbf{IJ} + v_1\,w_3\,\mathbf{IK} + v_2\,w_1\,\mathbf{JI} + v_2\,w_3\,\mathbf{JK} + v_3\,w_1\,\mathbf{KI} + v_3\,w_2\,\mathbf{KJ}$$

$$= v_1\,w_2\,\mathbf{K} - v_1\,w_3\,\mathbf{J} - v_2\,w_1\,\mathbf{K} + v_2\,w_3\,\mathbf{I} + v_3\,w_1\,\mathbf{J} - v_3\,w_2\,\mathbf{I}$$

$$= (v_2\,w_3 - v_3\,w_2)\,\mathbf{I} - (v_3\,w_1 - v_1\,w_3)\,\mathbf{J} + (v_1\,w_2 - v_2\,w_1)\,\mathbf{K}$$

According to Wikipedia, this is the distributive cross product. The coefficients can be written in vector form.

$$\begin{pmatrix} v_2\,w_3 - v_3\,w_2 \\ v_3\,w_1 - v_1\,w_3 \\ v_1\,w_2 - v_2\,w_1 \end{pmatrix}$$

<https://en.wikipedia.org/wiki/Cross_product>

We can now write $\mathbb{V}\mathbb{W}$ = *(vw -* ***V • W****) + (v* ***W*** *+ w* ***V*** *+* ***V × W****).*

(17) p. 292, *Sentence 1: As a simple check, note that the Möbius transformation corresponding to the Möbius matrix* ***K*** *is K(z) = -z. Make sure you can see why this is as it should be*.

We represent z as a 2x2 matrix based on z = $\frac{z+0}{0+1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = [z]. Then multiply by K, given at the bottom of p. 291.

$$\mathrm{K(z)} \equiv \mathbf{K}.[\mathrm{z}] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \equiv \frac{\mathrm{iz}}{-i} = -\mathrm{z}$$

(18) Now we can state the surprising connection with quaternions: under matrix multiplication, these binary rotation matrices obey [exercise] exactly the same laws (24) and (25) as Hamilton's I, J, K.

$$\boldsymbol{I}^2 = \boldsymbol{J}^2 = \boldsymbol{K}^2 = \mathbf{-1} \qquad (24)$$

$$\mathbf{IJ} = \mathbf{K} = -\mathbf{JI}, \quad \mathbf{JK} = \mathbf{I} = -\mathbf{KJ}, \quad \mathbf{KI} = \mathbf{J} = -\,\mathbf{IK} \qquad (25)$$

Substitute the values for the basis vectors from the bottom of p. 291. We write out the squares in full to make the mental calulations easier.

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} . \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

This shows that under matrix multipliction, the binary rotation matrices obey rule (24).

Test rule (25). We will test only the first one: **IJ** = **K** = -**JI**

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

This shows that the binary rotation matrices obey the first rule of (25).

(19), p. 292, ℙ 2: The general rotation matrix $[R_{\boldsymbol{v}}^{\psi}]$ in (21) can be expressed [exercise] as the quaternion

$$\mathbb{R}_{\boldsymbol{v}}^{\psi} = \cos(\psi/2) + \mathbf{V} \sin(\psi/2)$$

Substitute the full abstract form of V. Then substitute the abstract matrix forms of **1**, **I**, **J**, and **K** and expand the matrix sum.

$$\mathbb{R}_{\boldsymbol{v}}^{\psi} = \cos(\psi/2)\, \mathbf{1} + (v_1\, \mathbf{I} + v_2\, \mathbf{J} + v_3 \mathbf{K}) \sin(\psi/2)$$

$$= \cos(\psi/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v_1 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \sin(\psi/2) + v_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin(\psi/2) + v_3 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \sin(\psi/2)$$

$$= \begin{pmatrix} \cos(\psi/2) & 0 \\ 0 & \cos(\psi/2) \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{i}v_1 \sin(\psi/2) \\ \mathrm{i}v_1 \sin(\psi/2) & 0 \end{pmatrix} +$$
$$\begin{pmatrix} 0 & -v_2 \sin(\psi/2) \\ v_2 \sin(\psi/2) & 0 \end{pmatrix} + \begin{pmatrix} \mathrm{i}v_3 \sin(\psi/2) & 0 \\ 0 & -\mathrm{i}v_3 \sin(\psi/2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi/2) + \mathrm{i}v_3 \sin(\psi/2) & \mathrm{i}v_1 \sin(\psi/2) - v_2 \sin(\psi/2) \\ \mathrm{i}v_1 \sin(\psi/2) + v_2 \sin(\psi/2) & \cos(\psi/2) - \mathrm{i}v_3 \sin(\psi/2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi/2) + \mathrm{i}v_3 \sin(\psi/2) & (-v_2 + \mathrm{i}v_1)\, \mathrm{i}\sin(\psi/2) \\ (v_2 + \mathrm{i}v_1) \sin(\psi/2) & \cos(\psi/2) - \mathrm{i}v_3 \sin(\psi/2) \end{pmatrix}$$

The quaternion coefficients are all real numbers, so they can be taken as equivalent to the coefficients of the basis vectors of the unit vector in (21), where $\mathrm{l} \equiv v_1$, $\mathrm{m} \equiv v_2$, $\mathrm{n} \equiv v_3$.

$$\mathbb{R}_{\boldsymbol{v}}^{\psi} \equiv \begin{pmatrix} \cos(\psi/2) + \mathrm{in} \sin(\psi/2) & (-m + \mathrm{il}) \sin(\psi/2) \\ (m + \mathrm{il}) \sin(\psi/2) & \cos(\psi/2) - \mathrm{in} \sin(\psi/2) \end{pmatrix} = [R_{\boldsymbol{v}}^{\psi}] \qquad (21)$$

(20) p. 292, ℙ 2, last sentence: *Once again, but more easily than before, we deduce that this is the rotation of* $\psi = (2Pi/3)$ *about the axis* $\boldsymbol{v} = \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$.

In the previous equation showing the multiplication of the quaternions, the scalar part is written with “1” rather than “**1**”, but to calculate this problem, one has to substitute the 2x2 identity matrix, for which **1x1** = **1**. The sum then matches the right hand side of (22), p. 290. The deduction can be made more easily because adding matrices is easier than multiplying them.