Notes on Needham, Chapter 6, Section III Hyperbolic Geometry
G. Palmer, February 7, 2017, 6:30 pm, PST

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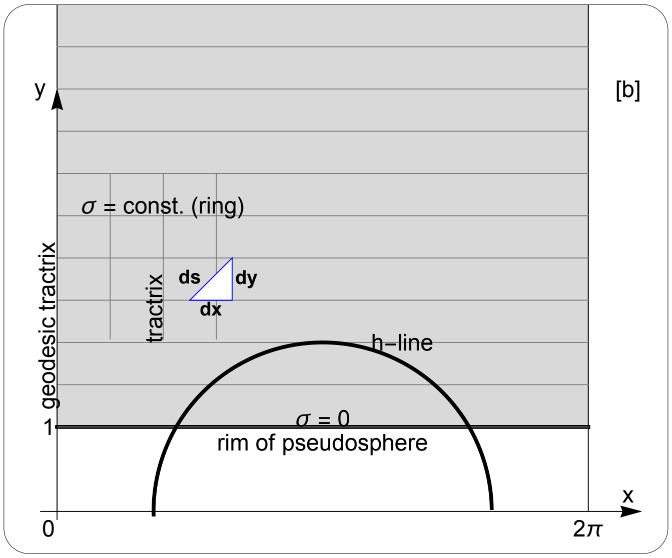

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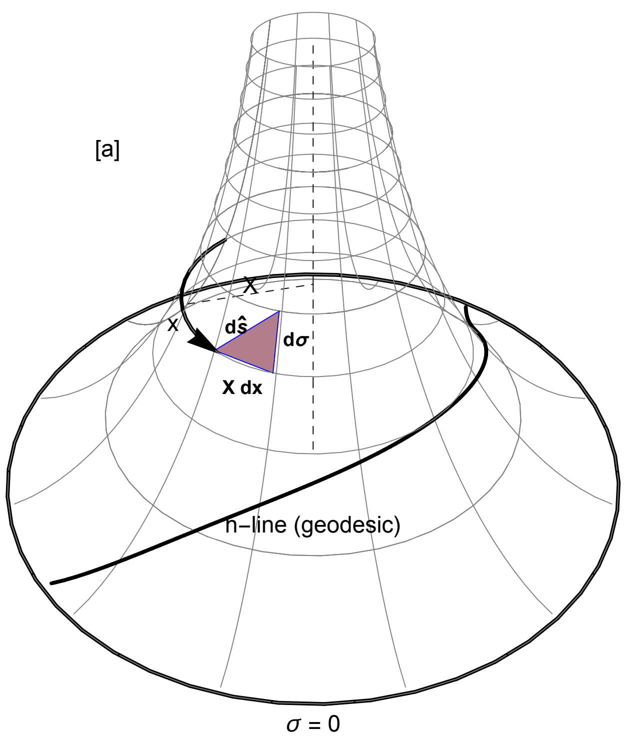

Figures [a] and [b] (After Figure [20], p. 297)

The point of this exercise is to plot the pseudosphere and objects on the pseudosphere from points in the map. It owes a lot to Vasco's comments on the forum. The function follows the principles outlined on pp. 296-298, but the text had to be supplemented with an equation from Wikipedia <https://en.wikipedia.org/wiki/Tractrix>. This function transforms a complex point in the map to an {X, Y, Z} point in the pseudosphere.

mapToPseudoSph(z) =
  If Im(z) ≥ 1,
    r = 1/Im(z);   (r corresponds to "X" in Figure [20]; i.e. X = 1/y)
    X = r Cos(Re(z));
    Y = r Sin(Re(z));
    $Z = \ln\left[\frac{1+\sqrt{1-r^2}}{r}\right] - \sqrt{1-r^2}$
    return {X, Y, Z}

The function mapToPseudoSph(z) was used to transform points from Figure [b]. The transformed points were used to plot Figure [a]. I would like to stress that all the objects in Figure [a] were plotted by transforming points of Figure [b]. Sometimes all the points were transformed, as with the points of rings and the h-line; sometimes just key or corner points were transformed, as with the triangle and the tractrix generator, which was rotated to form the illusion of a surface. Points where Im(z) < 1 were discarded.

END OF EXERCISE

p. 294, ℙ 1 *[exercise] ... The tractrix can be constructed as shown in [18b], namely as an orthogonal trajectory through the family of circles of radius R centred on the axis.*

By construction, one point p_{axis} lies on the axis and the other p_{contact} lies on the curve at a constant distance of R from p_{axis}. Since p_{axis} is fixed at the start, initial p_{contact} must lie somewhere on a circle with center p_{axis} and radius R.

p. 295, ℙ 2, *...It follows by symmetry [exercise] that*

$\tilde{r}$ = radius of curvature of the generating tractrix
r = the segment of the normal from the surface to the axis

I am unsure what is meant by "*by symmetry*". It might be a reference to p. 148, where we find the following:

If two points are symmetric with respect to a circle, then their images under a Möbius transformation are symmetric with respect to the image circle. This is called the "Symmetry Principle."

How could this be applied to Figure [19a] on p. 295? Let p be the point at the intersection of the ray from O with the tractrix. Call the large dashed circle on the right K. Call the dashed circle within the tractrix on the left C. It appears the planes of K and C are orthogonal. Let p be the point where K and C touch. It is clear that $I_K(K)$ replaces K. C has its center at $(r + \tilde{r})\, e^{i\theta}$. Calculation using the center of C and p = $\tilde{r}\, e^{i\theta}$ shows that $I_C(C)$ replaces C. So $I_K(p) = I_C(p)$ = p. This is the only point in the tractrix where the equation is true for this K and this C. The radius of the circle of curvature K is $\tilde{r}$. The radius of the circle of curvature C is r.

It follows from this that

$\tilde{r}$ = radius of curvature of the generating tractrix
r = the segment of the normal from the surface to the axis

p. 297, Suggested rewrite:

Regarding Ch 6, ℙ III, subsection 3, A Conformal Map of the Pseudosphere, paragraph 2, p. 297 seems very hard to follow. I suggest a paraphrase for the remainder of the paragraph beginning at sentence 4 immediately following "...separated by distance dx".

"X dx on the pseudosphere corresponds to dx on the map and the actual distance X ds on the pseudosphere corresponds to apparent distance ds on the map. In other words, the metric is

$$d\hat{s} = X\ ds."$$

See p. 284 for definitions of $d\hat{s}$ and ds. The term "corresponds" does not mean that two corresponding expressions are equal, only that one is a transformation of the other.

Vasco (Forum, Sept 28) has rightly objected that Needham was 'appealing to the ideas of the previous two chapters, in particular the "amplitwist" concept' and attempting to prove $d\hat{s}$ = X ds, so I could not assume it. So I retract the suggestion. But I would like to suggest an addition to the end of the clause "an infinitesimal line-segment emanating from (x,σ) in any direction must be multiplied by the same factor $(1/X) = e^{\sigma}$". I would like to add "to obtain its value on the map".

[Exercise], p. 300, paragraph 2, line 2

Suppose you are walking at a steady speed of ln 2. ... Integration of (dy/y) shows [exercise] that you reach y = 1 after one unit of time, y = (1/2) after two units of time, y = (1/4) after 3 units of time, etc. ...

The subject is walking on the hyperbolic plane at a steady speed of ln(2). On the h-plane, infinitesimal changes in distance are represented as $d\hat{s}$. On the map, a doppleganger of the subject is reducing the y-distance from the horizon by half with each unit of time. We are given $d\hat{s} = \frac{ds}{y}$ and $\frac{d\hat{s}}{dt}$ = -ln(2). In this particular case, the subject is walking down the tractrix (p. 299, last paragraph), so $\frac{ds}{y} = \frac{dy}{y}$

We can write

$$\frac{dy}{y} = -\ln(2)\ dt$$

Integrating both sides

ln(y) = -ln(2) t + c (constant of integration)

At the start, y = 2, t = 0, so c = ln(2).

Now we can write ln(y) = -ln(2) t + ln(2)

$y = e^{-\ln(2)\,t + \ln(2)}$

Then tablulate for values of t.

t	y	y
1	e^0	1
2	$e^{-2\ln(2)+\ln(2)}$	1/2
3	$e^{-3\ln(2)+\ln(2)}$	1/4
...		

The language pertaining to the hyperbolic plane and the map seems ambiguous and invites confusion:

p. 301, "an 'h-line' will mean a 'hyperbolic line', while a 'line' will refer to an ordinary straight line in the map", but on p. 302, ℙ 1, "vertical lines in the map should also be geodesic, i.e. they should be examples of h-lines."

Thus it is not immediately clear whether (34), referring to [22a], points to the map or to the hyperbolic plane. In this case, the gray background and the symbols ds and y indicate the map.

In (36), it becomes clear that h-line (geodesic in the hyperbolic plane) can refer to a line in the map, but only certain lines in the map are h-lines: half-lines and semi-circles orthogonal to the horizon.

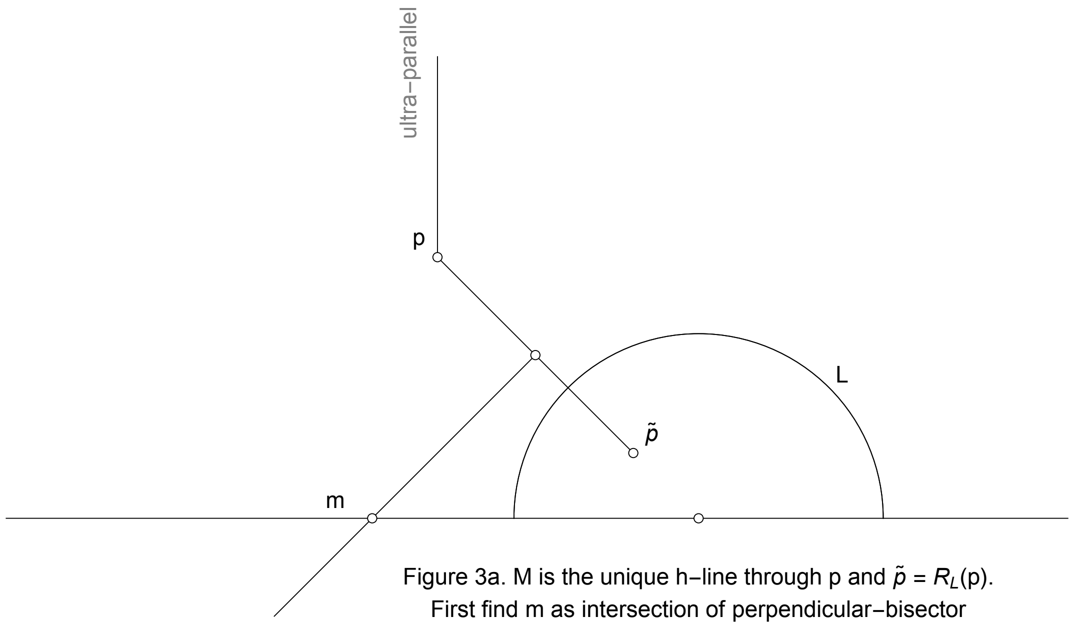

Figure 3a. M is the unique h−line through p and $\tilde{p} = R_L(p)$.
First find m as intersection of perpendicular−bisector
of $p\tilde{p}$ with horizon.

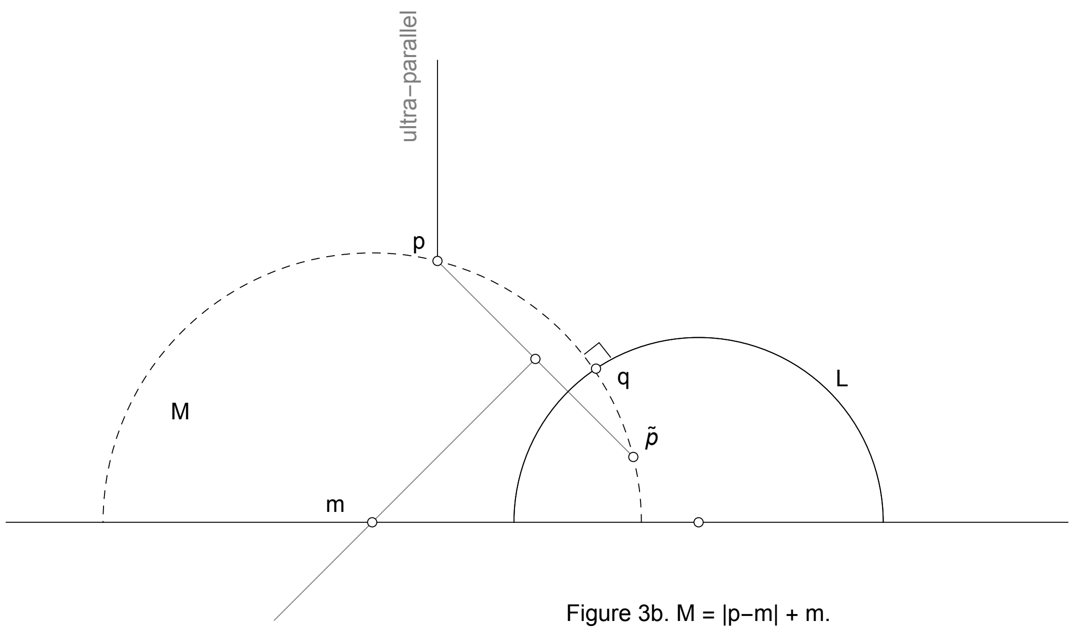

Figure 3b. M = |p−m| + m.

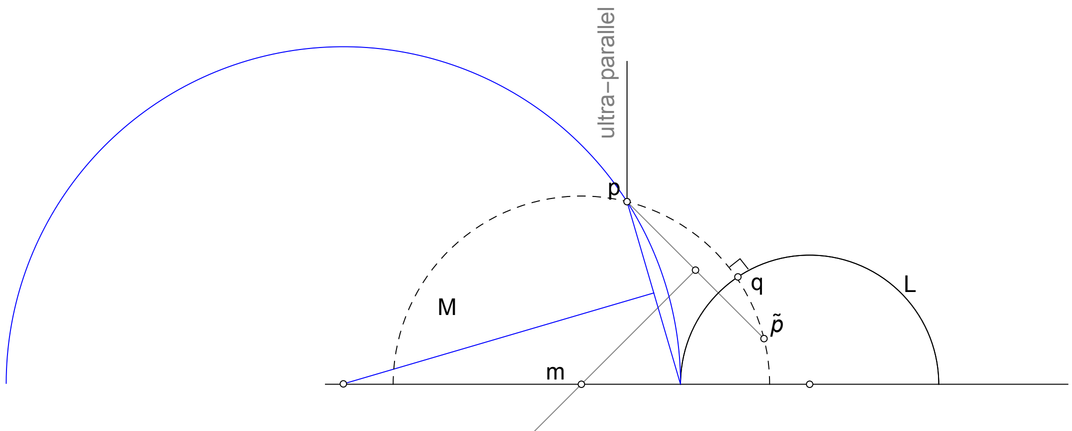

Figure 3c. Asymptotic added.

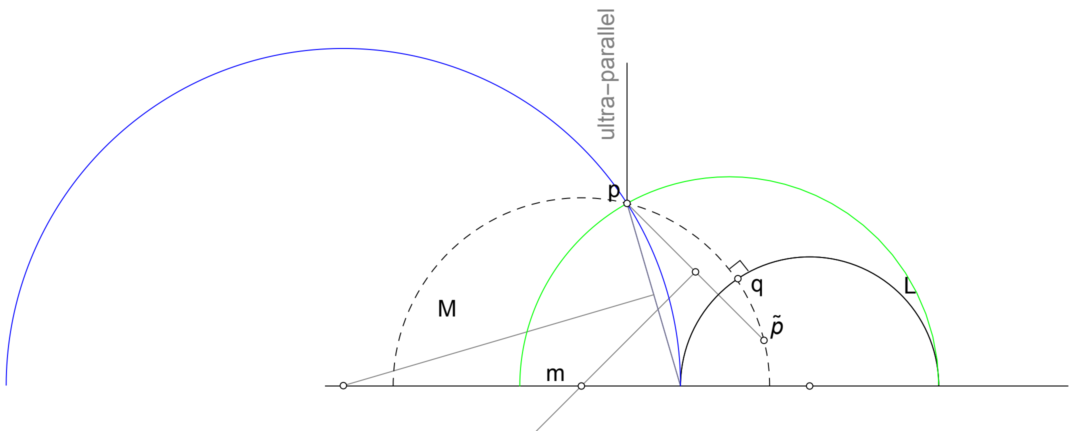

Figure 3d. Asymptotic line inverted: Blue to Green.

[Exercise], p. 305, $\mathbb{P}$ 2, line 2

Any circle drawn through p and $\mathcal{R}_L(p)$ is orthogonal to L (p. 129, quote under Figure [6]). To draw a circle through p and $\mathcal{R}_L(p)$, construct the perpendicular bisector of p$\tilde{p}$, where $\tilde{p} = \mathcal{R}_L(p) = \mathcal{I}_L(p)$ (Figure 3a). Solve for m as the intersection of the bisector and the horizon. Plot M = |p-m|. M is orthogonal to L and it is the only h-line passing through p that cuts L at right angles (Figure 3b).

[Exercise], p. 305, $\mathbb{P}$ 3, line 2

Let M be a circle of inversion. Then $\mathcal{I}_M$(M) = M, so $\mathcal{I}_M$(p) = p. L in [24] is orthogonal to M, so $\mathcal{I}_M$ swaps the end points of L (the points of intersection of L and the line of the horizon). Now we see that the asymptotic lines are inversions in M at p and at the endpoints of L on the horizon. Inversion is anticonformal, so it preserves the magnitude but not the orientation of angles. If we apply $\mathcal{I}_M$ to either asymptotic h-line (Figures 3c and 3d), it is swapped with the other and the magnitude of the angle of parallelism is preserved (i.e. carried over to the new asymptotic), so the two angles are equal. Hence, *M bisects the angle at p contained by the asymptotic lines.*

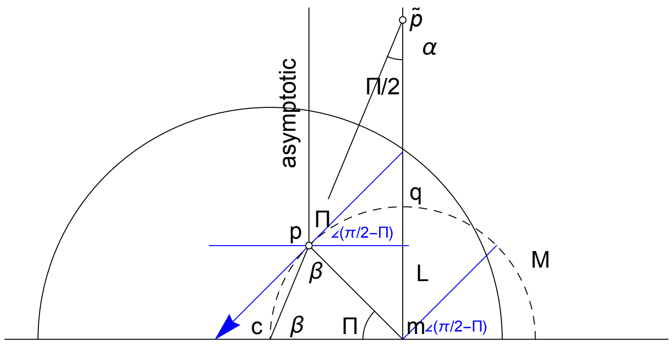

Figure 4.

[Exercises], p. 306, ℙ 3. *From the fact that the radius pm is orthogonal to the circle M it follows [exercise] that the angle pmc equals Π.*

Answer: The angle between the asymptotic and the h-line segment pq is Π. Draw a horizontal line through p. Use the orthogonality of the tangent to plot a parallel to the tangent through m. The angle of this parallel with the horizon is (π/2 - Π). Then the angle pmc is (π - (π/2 + (π/2 - Π)) = π - π/2 - π/2 + Π = Π.

It then follows [exercise] that the angle $c\hat{p}m$ equals (Π/2), as illustrated.

Let α be the angle $c\tilde{p}m$. Let β be the angle cpm.

$2\,\beta = \pi - \Pi$ *by inspection of the isosceles triangle cpm*

$$\beta = \frac{\pi - \Pi}{2}$$

$\beta + \alpha = \text{Pi}/2$ by *inspection of the triangle $c\tilde{p}m$*

$$\alpha = \pi/2 - \beta = \pi/2 - \frac{\pi - \Pi}{2} = \Pi/2$$

The Three Types of Direct Motion

[Exercise], p. 307

$\mathcal{I}_C(z) = \frac{q\,\bar{z} + (R^2 - q^2)}{\bar{z} - q}$, where q is on the real axis

Composing two such functions, we find [exercise] that a motion of type M corresponds to a Möbius transformation

$M(z) = \frac{az+b}{cz+d}$, where a, b, c, d are real, and (ad - bc) > 0.

Let $I_S(z) = \frac{s\,\bar{z}+(R_s{}^2-s^2)}{\bar{z}-s}$, where s is the center of circle S on the real axis and R_s is the radius of S.

$$M(z) = I_S(\mathcal{I}_C(z)) = I_S\left(\frac{q\,\bar{z}+(R^2-q^2)}{\bar{z}-q}\right) = \frac{s\overline{\left(\frac{q\,\bar{z}+(R^2-q^2)}{\bar{z}-q}\right)}+(R_s{}^2-s^2)}{\overline{\left(\frac{q\,\bar{z}+(R^2-q^2)}{\bar{z}-q}\right)}-s} = \frac{s\,\frac{qz+(R^2-q^2)}{z-q}+(R_s{}^2-s^2)}{\frac{qz+(R^2-q^2)}{z-q}-s}$$

$$= \frac{s(qz+(R^2-q^2))+(R_s{}^2-s^2)\,(z-q)}{qz+(R^2-q^2)-s(z-q)} = \frac{sqz+s(R^2-q^2))+(R_s{}^2-s^2)\,(z-q)}{qz+(R^2-q^2)-sz+qs}$$

$$= \frac{(qs+R_s{}^2-s^2)\,z+s(R^2-q^2))-q(R_s{}^2-s^2))}{(q-s)\,z+(qs+R^2-q^2)}$$

This has the format $\frac{az+b}{cz+d}$, where

$a = R_s{}^2 + s(q-s)$

$b = s\,(R^2-q^2) - q\,(R_s{}^2-s^2)$

$c = q-s$

$d = R^2- q(q-s)$

R, R_s, q, and s are real, so a, b , c, and d are real and we require that (ad-bc) be > 0. We use *Mathematica* to expand ad - bc = $R^2\,R_s{}^2$, so ad - bc > 0.

```
a = Rs^2 + s (q - s) ;

b = s (R^2 - q^2) - q (Rs^2 - s^2);

c = q - s;

d = R^2 - q (q - s) ;

Print["ad-bc = ", (a d - b c) // Expand];
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ad-bc = R^2 Rs^2
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Exercise, Subsection 10, The Poincaré Disc, p. 316, ℙ1, line -1

...Note that this implies [exercise] that the h-lines of the disc are precisely the images of h-lines in the upper half-plane.

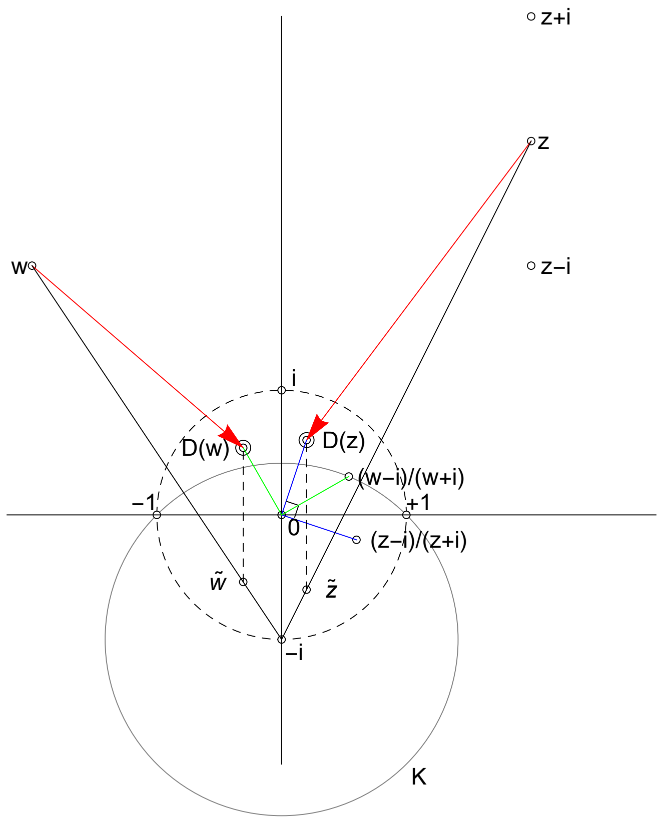


Figure 5. The Poincaré Disc

Exercise, Subsection 10, The Poincaré Disc, p. 316, bottom line

Finally, recalling that a Möbius transformation is uniquely determined by its effect on three points, and noting that ±1 remain fixed, we deduce [exercise] that

$$D(z) = \frac{iz+1}{z+i}$$

If z = 1, then $k\frac{z-i}{z+i} = 1 \Longrightarrow k\frac{1-i}{1+i} = 1 \Longrightarrow k = \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i-1}{1+1} = i$

So $D(z) = i\frac{z-i}{z+i} = \frac{iz+1}{z+i}$

Similarly, if z = -1, then $k\frac{-1-i}{-1+i} = -1 \Longrightarrow k = \frac{-(-1+i)}{-1-i} = \frac{(1-i)^2}{(-1-i)(1-i)} = \frac{1-2i+1}{-1-1} = i$

D(z) $\frac{iz+1}{z+1}$ is equivalent to rotating the line (z-i)/(z+i) by Pi/2 or multplying (z-i)/(z+i) by i, as we can see in the figure by looking at the blue lines (or the green lines for (w-i)/(w+i)). How does this exemplify a unique effect on three points?

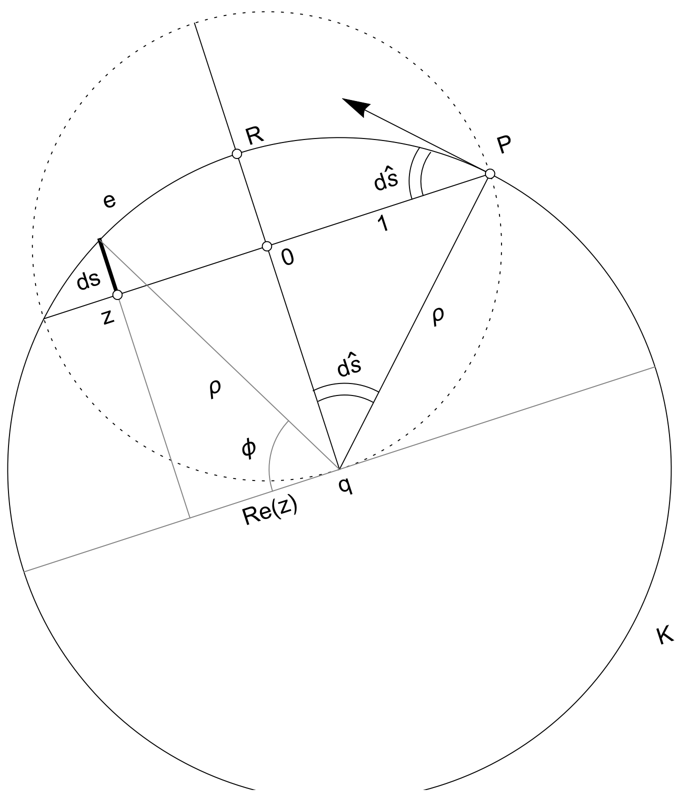
Alternatively, this may be derived by brute force [exercise] using the formula for inversion, (4), p. 125.

The brute force does not seem very brutish. The first sentence of this paragraph on p. 316 has D(z) as the composition z ↦ $\mathcal{I}_K$(z) and z ↦ $\overline{z}$. The composition can be written:

$$D(z) = \text{Conjugate}\left(\frac{2}{\text{Conjugate}(z-q)} + q\right) = \frac{2}{z-q} + \overline{q}$$

$$= \frac{2}{z+i} + i = \frac{2+(z+i)\,i}{z+i} = \frac{2+iz-1}{z+i}$$

$$= \frac{iz+1}{z+i}$$



At the bottom of p. 317, the last sentence stops abruptly without a period. It took me some time to realize that it resumes directly under the label for Figure [36] with $\rho\, d\hat{s} = 1$. So I would like to call that to the attention of others.

The last sentence contains "[draw a picture!]", but the equation itself, $\rho\, d\hat{s} = 1$, should probably have been labeled "[exercise]". The figure appears to be analogous to the polar form of differentiation illustrated in Figure [2], p. 218, where the orthogonal vector to the ray equals the angle of rotation times the radius, which is |z|. In our case, we are only interested in the absolute values of ρ and $d\hat{s}$. We add labels to points R and P as in Vasco's version. The length s of the arc RP equals ρ times the angle RqP, or s = $\rho\, d\hat{s}$. As ρ tends toward ∞, $d\hat{s} \to 0$, $\rho\, d\hat{s} \to 1$, so we write $\rho\, d\hat{s} = 1$. For K, we used ρ =

sqrt(2) and q = -i. This seemed as good as any, though we made no use of inversion in K. Vasco's version illustrates the change in $d\hat{s}$ with change in ρ.

If we had been working with the Poincaré upper half plane, we would have had to use the equation $d\hat{s}$ = ds/y = ds/Im(z), but we could not drop z below the real line as happens in the Poincaré disc. But with the disc, we see in (44) that $d\hat{s}$ depends only on ds and the absolute value of z. The point z can lie below the real line and its absolute value remains constant under rotation about a point, because z is centred at r $e^{i\theta}$ in the Poincaré disc (p. 317, ℙ -2). This plot was rotated after construction.