Notes on Σ, ℂ and Quaternions, pp. 188-289.
G. Palmer, September 8, 2016, 10:30 AM PST

The section on spherical geometry and quaternions offers ample opportunities for confusion. Some of the notation pertains to points on a sphere given in vector format, some to complex numbers and Möbius transformations on the complex plane ℂ. Please note that all matrix brackets should really be square brackets.

$R_{\hat{a}}^{\psi}$ refers to a rotation about the point $\hat{a}$ on the unit sphere Σ. The point $\hat{a}$ could be given as a multiple of a unit vector or by polar coordinates. In this chapter, it should probably be thought of as a unit vector because $\hat{a}$ appears as the end point of a vector on the unit sphere. The rotation $R_{\hat{a}}^{\psi}$ on Σ implies another rotation on ℂ written as $[R_a^{\psi}]$, where the complex number a on ℂ is the stereographic projection of the vector point $\hat{a}$. The square brackets around $[R_a^{\psi}]$ signify that this rotation is written in matrix format, so it can be multiplied with other matrices of Möbius transformations. An alternative way of writing such a rotation is the Möbius transformation

$$\frac{\mathrm{A}z+B}{-\overline{B}z+\overline{A}},$$

which, by stereographic projection, induces the most direct general motion of Σ. So we are working back and forth between ℂ and Σ with stereographic projection as the conformal function between the plane and the sphere. In Σ we have points, rotations on points, and rotations on unit vectors. In ℂ, we have complex numbers, Möbius transformations, and rotations written in 2x2 matrices.

In the unit sphere Σ, our notation for the points is

$$\hat{a} \equiv \mathbf{v} = \mathrm{l}\mathbf{i} + \mathrm{m}\mathbf{j} + \mathrm{n}\mathbf{k}$$

so **v** is a unit vector. These boldface lower case basis vectors can easily be confused with the quaternion basis vectors, which have a different meaning and algebra. Then the a induced in ℂ by $\hat{a}$ becomes

$$\mathrm{a} = \frac{l+\mathrm{im}}{1-n} \text{ and } |a|^2 = \frac{1+n}{1-n}$$

There is another rotation $[R_{\mathbf{v}}^{\psi}]$, which is the matrix form of a Möbius transformation induced by the rotation of the angle ψ about **v**, which is a unit vector with it's endpoint in Σ. This notation seems at variance with the notation $[R_a^{\psi}]$, in which the subscripted a is a complex number. In $[R_{\mathbf{v}}^{\psi}]$, the matrix and the result are in ℂ as with $[R_a^{\psi}]$, but the vector **v** is in Σ. The subscripted **v** is a unit vector with coefficients l, m, and n, so the matrix form of the Möbius transformation corresponding to the binary transformation ($\psi = \pi$) is

$$[R_{\mathbf{v}}^{\pi}] = \begin{pmatrix} \mathrm{in} & -m+\mathrm{il} \\ m+\mathrm{il} & -\mathrm{in} \end{pmatrix} = [R_a^{\pi}],$$

where a is the complex point induced by $\hat{a}$, the endpoint of **v**.

Quaternions provide what is really a third system or field that can be related algebraically to both the unit vector and point representations in Σ and the 2 x 2 matrices in $\mathbb{C}$. The vector part **V** of a quaternion corresponds to the unit vector restricted to Σ, but it has its own protocols for multiplication. Needham introduced the convention of writing quaternion basis vectors **1**, **I**, **J**, and **K** as 2x2 matrices for the binary transformation ($\psi = \pi$), as though they were Möbius transformations. These matrices obey the same rules (24, 25) as the quaternion basis vectors.