Needham, Chapter 6, Exercise 9
G. Palmer, November 28, 2016, 9:15 pm, PST

9 *(i) Figure [40a] gave a two-step process for carrying a point z in the Poincaré disc (PD) to the corresponding point z' in the Klein model. Explain why the net mapping z ↦ z' of the disc to itself is the one shown in figure [a] below, where C is an arbitrary circle passing through z and orthogonal to the unit circle U.*

Figure [a] appears to present the shadow of the Klein hemisphere model (shown in [40a]) from above. The semi-circle on the sphere appears as a straight line on the PD. The line of projection from S in the sphere to a point $\tilde{z}$ on the semicircle in the upper hemisphere appears as the line from 0 to z' when projected onto the PD. The line projected vertically from $\tilde{z}$ down to z' appears as the point z' in the PD.

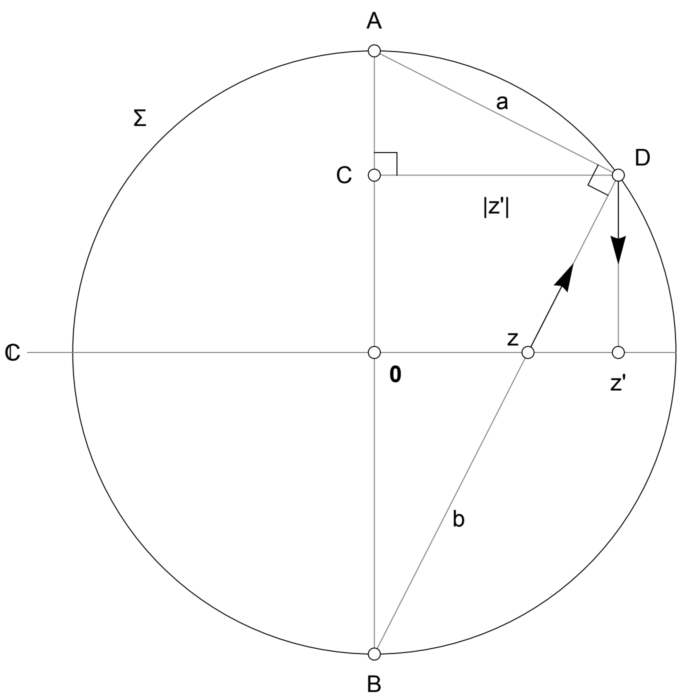

Figure 1. Facsimile of [b], Ex. 9, p. 331

(ii) Figure [b] is a vertical cross section of [40a] through z and z'. Deduce that

$$\frac{|z'|}{a} = \frac{1}{b} \quad \textit{and} \quad \frac{a}{|z|} = \frac{2}{b}.$$

In Figure 1, add labels A, B, C, D. Note that b = [Bz]. Figure 1 shows

Triangles ACD and ABD are similar because they share a side (AD) and a right angle.
Triangle B0z is similar to ACD and ABD because it shares two angles with CBD. Therefore

$$\frac{|z'|}{a} = \frac{1}{b} \quad \text{(by similarity of ACD and B0z)}$$

$$\frac{a}{|z|} = \frac{2}{b} \quad \text{(by similarity of ABD and B0z)}$$

By multiplying these two equations, deduce that $z' = \frac{2z}{1+|z|^2}$. *[Thus we have a geometric explanation of the result (20), p. 146.*

$$b^2 = 1 + |z|^2 \text{ by the Pythagorean theorem}$$

Multiply the equations:

$$\frac{|z'|}{a}\,\frac{a}{|z|} = \frac{1}{b}\,\frac{2}{b} \Longrightarrow b^2 = \frac{2\,|z|}{|z'|}$$

Then

$$1 + |z|^2 = \frac{2\,|z|}{|z'|}$$

$$|z'| = (2\,|z|)/(1 + |z|^2)$$

Since z and z' fall on the real line, we can write this:

$$z' = 2z/(1 + |z|^2)$$

Out[1666]=

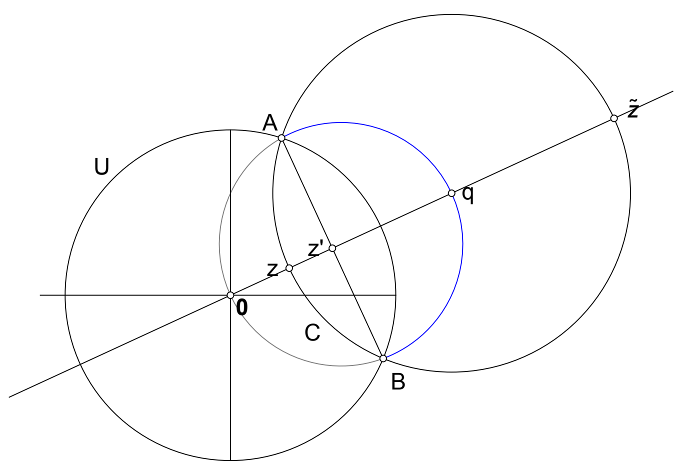

Figure 2. Elaboration of [a], Ex. 9, p. 331

(iii) This formula can be derived directly from figure [a] without the assistance of the hemisphere. Redraw the figure with C chosen orthogonal to 0z. Explain geometrically why the centre of C may be viewed as either $\mathcal{I}_U(z')$, *or as the midpoint of z and* $\mathcal{I}_U(z)$. *Conclude that*

$$1/\overline{z'} = \mathcal{I}_U(z') = \tfrac{1}{2}[z + I_U(z)] = \tfrac{1}{2}[z + 1/\overline{z'}],$$

from which the result follows immediately. [i. e. z' = $2z/(1+|z|^2)$]

Explain geometrically why the centre of C may be viewed as $\mathcal{I}_U(z')$.

We make a limited analogy with Figure [38], but note that C in Figure 2 corresponds to J in [38] and U in Figure 2 corresponds to C in [38], Chapter 3, Section IX, Subsection 3 *Interpreting the Simplest Formula Geometrically,* pp. 178-179. We use only the first part of the argument prior to completing construction of the Möbius transformation.

By analogy with the argument accompanying Figure [38], we are looking for the circle C with centre q such that $\mathcal{I}_C$ swaps z' and 0. Since z' and 1/z' are symmetric with respect to U, their images under $\mathcal{I}_C$ are symmetric with respect to $\mathcal{I}_C$(U) = U. Because we want $\mathcal{I}_C$(z') = 0, we deduce that $\mathcal{I}_C(1/\overline{z'}) = \infty$ (which is the ultimate extension of the straight line AB in Figure 2). But the point mapped to infinity is the center of C, so q = $1/\overline{z'}$.

From another perspective, $\mathcal{I}_U$(AB) (blue) is an arc of a circle passing through q. By symmetry, this arc must also be an arc of $\mathcal{I}_C$(Ext(AB)), where Ext(AB) refers to the extension of AB in both directions to infinity. Since z' is the midpoint of AB, and z' is a point in the line 0q, and q is the point $\mathcal{I}_C(\infty)$, q must be $\mathcal{I}_U$(z').

...or as the midpoint of z and $\mathcal{I}_U(z)$.

Since C is orthogonal to U, inversion of C in U maps C to C. z and $\tilde{z}$ are opposite points in C by construction, so q is the midpoint of z and $\tilde{z}$.

Conclude that

$$1/\overline{z'} = \mathcal{I}_U(z') = \frac{1}{2}[z + I_U(z)] = \frac{1}{2}[z + 1/\overline{z}]$$

We have deduced that q = $\mathcal{I}_U$(z'), and we know that $I_U(z) = 1/z$, so $\mathcal{I}_U(z') = 1/\overline{z'}$ = q. We have deduced that q lies at the midpoint of *z and* $\mathcal{I}_U$(z), so

$$q = \frac{1}{2}\,(\,I_U(z) - z) + z = \frac{1}{2}\,(\,I_U(z) - z) + \frac{2z}{2} = \frac{1}{2}[\,z\,+\,I_U(z)]$$

And since $I_U(z) = 1/\overline{z}$, q = $\frac{1}{2}[\,z\,+\,1/\overline{z}]$.

Conclusion: $1/\overline{z'} = \mathcal{I}_U(z') = \frac{1}{2}[z + I_U(z)] = \frac{1}{2}[z + 1/\overline{z}]$

from which the result follows immediately.

$$1/\overline{z'} = \frac{1}{2}[z + 1/\overline{z}] \Longrightarrow \overline{z'} = \frac{2}{z+1/\overline{z}}$$

$$z' = \frac{2}{\overline{z}+1/z} = \frac{2z}{z\,\overline{z}+1} = (2z)/(1+|z|^2)$$