Needham, Chapter 6, Exercise 7, p. 329.
G. Palmer, September 21, 2016, 8:30 AM, PST

7. The following proof of (29) is based on a paper of H.S.M. Coxeter [1946].

(i) Use (27) to show that the pure quaternions $\mathbb{P}$ and $\mathbb{A}$ are orthogonal if and only if $\mathbb{P}\,\mathbb{A} + \mathbb{A}\,\mathbb{P} = 0$.

$$\mathbb{V}\,\mathbb{W} = -\mathbf{V} \bullet \mathbf{W} + \mathbf{V} \times \mathbf{W} \qquad (27)$$

If $\mathbb{P}$ and $\mathbb{A}$ are pure quaternions, then if $\mathbb{P}\,\mathbb{A} + \mathbb{A}\,\mathbb{P} = 0$, then so are their vector parts **P** and **A**. Then, by (27)

$$-\mathbf{P} \bullet \mathbf{A} + \mathbf{P} \times \mathbf{A} + (-\mathbf{A} \bullet \mathbf{P} + \mathbf{A} \times \mathbf{P}) = 0$$

$$\Longrightarrow -2\, -\mathbf{P} \bullet \mathbf{A} + (\mathbf{P} \times \mathbf{A} - \mathbf{P} \times \mathbf{A}) = 0 \qquad \text{because } A \times B = -B \times A$$

$$\Longrightarrow 2\,\mathbf{P} \bullet \mathbf{A} = 0$$

$$\Longrightarrow \mathbf{P} \bullet \mathbf{A} = 0$$

This shows that that $\mathbb{P}$ and $\mathbb{A}$ are orthogonal if $\mathbb{P}\,\mathbb{A} + \mathbb{A}\,\mathbb{P} = 0$.

If $\mathbb{P}\,\mathbb{A} + \mathbb{A}\,\mathbb{P} \neq 0$, then $(\mathbf{P} \times \mathbf{A} + \mathbf{A} \times \mathbf{P}) \neq 0$ or $\mathbf{P} \bullet \mathbf{A} \neq 0$. $(\mathbf{P} \times \mathbf{A} + \mathbf{A} \times \mathbf{P})$ is always equal to zero by the definition of cross products. **P** and **A** are pure quaternions by definition. If $\mathbf{P} \bullet \mathbf{A} \neq 0$, then $\mathbb{P} \bullet \mathbb{A} \neq 0$, showing that they are not orthogonal, so $\mathbb{P}$ and $\mathbb{A}$ are orthogonal only if $\mathbb{P}\,\mathbb{A} + \mathbb{A}\,\mathbb{P} = 0$.

(ii) If $\mathbb{A}$ has unit length, so that $\mathbb{A}^2 = -1$, deduce that the previous equation may be expressed as $\mathbb{P} = \mathbb{A}\,\mathbb{P}\,\mathbb{A}$.

$$\mathbb{P}\,\mathbb{A} + \mathbb{A}\,\mathbb{P} = 0$$

$$\mathbb{A}\,\mathbb{P} = -\mathbb{P}\,\mathbb{A}$$

$$\mathbb{A}\,\mathbb{A}\,\mathbb{P} = -\mathbb{A}\,\mathbb{P}\,\mathbb{A}$$

$$-\mathbb{P} = -\mathbb{A}\,\mathbb{P}\,\mathbb{A}$$

$$\mathbb{P} = \mathbb{A}\,\mathbb{P}\,\mathbb{A}$$

*(iii) Now keep the pure, unit quaternion $\mathbb{A}$ fixed, but let $\mathbb{P}$ represent an arbitrary pure quaternion. Let Π_A denote the plane with normal vector **A** that passes through the origin, so that its equation is $\mathbf{P} \bullet \mathbf{A} = 0$. Now consider the transformation*

$$\mathbb{P} \mapsto \mathbb{P}' = \mathbb{A}\,\mathbb{P}\,\mathbb{A} \qquad (49)$$

Show that (a) $\mathbb{P}'$ is automatically pure, and $|\mathbb{P}'| = |\mathbb{P}|$, so that (49) represents a motion in space; (b) every point on Π_A remains fixed; (c) every vector orthogonal to Π_A is reversed. Deduce that (49) represents reflection $\mathcal{R}_{\Pi_A}$ of space in the plane Π_A.

(a) *Show that $\mathbb{P}'$ is automatically pure.*

$\mathbb{A}$ and $\mathbb{P}$ are pure quaternions by construction. Therefore,

$\mathbb{P}' = \mathbf{A\ P\ A}$

$\mathbf{= (- A \bullet P + A} \times \mathbf{P)\ A}$

$\mathbf{= (A} \times \mathbf{P)\ A}$

The cross product **A × P** of two pure orthogonal quaternions is a pure quaternion because it has no dot product, and A is a pure quaternion. The vector **A × P** is orthogonal to both **A** and **P**, so **(A × P) • A** = 0**.** Then

$\mathbf{(A \times P)\ A} = -\ \mathbf{(A \times P) \bullet A + (A \times P)} \times \mathbf{A = (A \times P)} \times \mathbf{A}$

(A × P) × **A** is a pure quaternion equivalent to **V** sin(ψ/2) for some **V** and some ψ**.** Hence, $\mathbb{P}'$ is automatically a pure quaternion.

and $|\mathbb{P}'| = |\mathbb{P}|$**:**

Rewrite **A P A**. $\mathbf{A} = \mathbb{R}_{\boldsymbol{A}}^{\psi}$, because $|\mathbf{A}| = 1$, and $\mathbb{P} = |\mathbb{P}|\mathbb{R}_{\mathbf{P}}^{\theta}$.

$\mathbb{P}' = \mathbb{R}_{\boldsymbol{A}}^{\psi}\ |\mathbb{P}|\mathbb{R}_{\mathbf{P}}^{\theta}\ \mathbb{R}_{\boldsymbol{A}}^{\psi}$

$= |\mathbb{P}|\mathbb{R}_{\boldsymbol{A}}^{\psi}\mathbb{R}_{\mathbf{P}}^{\theta}\ \mathbb{R}_{\boldsymbol{A}}^{\psi}$ (1)

(1) shows that $|\mathbb{P}'| = |\mathbb{P}|$ because $\mathbb{R}_{\boldsymbol{A}}^{\psi}\mathbb{R}_{\mathbf{P}}^{\theta}\ \mathbb{R}_{\boldsymbol{A}}^{\psi}$ signifies only rotation.

(b) every point on Π_A remains fixed;

We have shown that $\mathbb{P}' = |\mathbb{P}|\mathbb{R}_{\boldsymbol{A}}^{\psi}\mathbb{R}_{\mathbf{P}}^{\theta}\ \mathbb{R}_{\boldsymbol{A}}^{\psi}$. Since $\mathbb{R}_{\boldsymbol{A}}^{\psi}$ represents a pure, unit quaternion as well as a rotation, we can write it $\mathbb{R}_{\boldsymbol{A}}^{\pi}$. Suppose that a point $\hat{p}$ lies in the vector P = X**i** + Y**j** + Z**k** in Π. We can imagine that $\hat{p}$ is on Σ, but it doesn't matter if it is. First, we rotate $\hat{p}$ by π. Then any rotation around **P** in $\Pi_{\boldsymbol{A}}$ will leave $\hat{p}$ fixed at the same distance from the origin. Then $\hat{p}$ undergoes the second rotation by π, which returns it to its original location. Hence, every point in Π_A remains fixed.

(c) every vector orthogonal to Π_A is reversed. Deduce that (49) represents reflection $\mathcal{R}_{\Pi_A}$ of space in the plane Π_A.

Check to see if $\mathbb{A}$ is reversed:

$$\mathbb{A}' = \mathbb{A}\ \mathbb{A}\ \mathbb{A}$$

$$= -\mathbb{A} \quad \text{by Ex. 6 (v)}$$

$\mathbb{A}$ is reversed. Let $\mathbb{B}$ be a vector $\mathbb{B} = \alpha\mathbb{A}$, where α is a real number. This vector could be offset from $\mathbb{A}$ by some quaternion $\mathbb{Q}$ in Π_A, which is therefore orthogonal to $\mathbb{A}$ (so $\mathbb{Q} = \mathbb{A}\mathbb{Q}\mathbb{A}$ by part (b)). Then we can write $\mathbb{B}' = \mathbb{A}\ (\mathbb{B}+\mathbb{Q})\ \mathbb{A} = \mathbb{A}\ (\alpha\mathbb{A} + \mathbb{Q})\ \mathbb{A} = (-\alpha + \mathbb{A}\mathbb{Q})\mathbb{A} = -\alpha\mathbb{A} + \mathbb{Q}$, so $\mathbb{B}$ is reversed at $\mathbb{Q}$, showing that every vector orthogonal to Π_A is reflected across Π_A, representing a reflection $\mathcal{R}_{\Pi_A}$ of space in the plane Π_A.

Out[1047]=

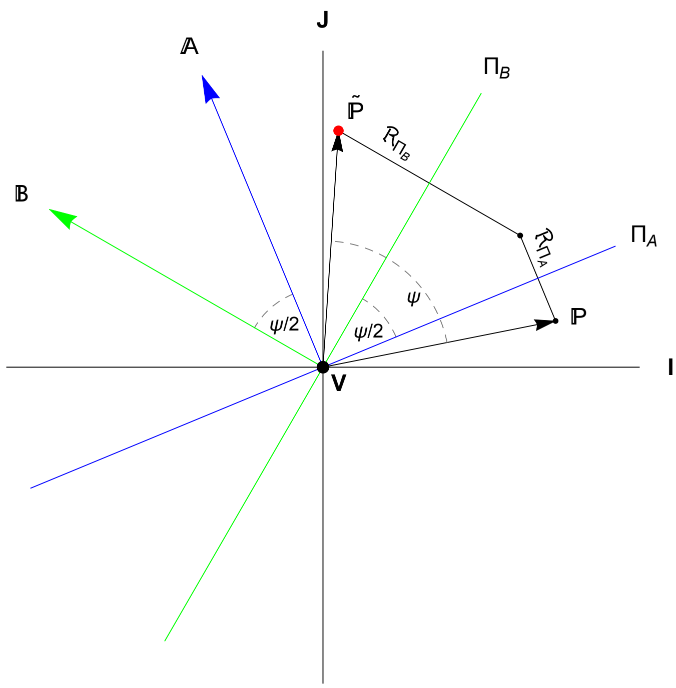

Fig 1. $\tilde{\mathbb{P}} = (\mathbb{B}\mathbb{A})\mathbb{P}(\mathbb{A}\mathbb{B}) = (\mathcal{R}_{\Pi_A} \circ \mathcal{R}_{\Pi_B})$

*(iv) Deduce that if the angle from Π_A to a second plane Π_B is $(\psi/2)$, and the unit vector along the intersection of the planes is **V**, then the rotation $\mathcal{R}_{\boldsymbol{V}}^{\psi} = (\mathcal{R}_{\Pi_B} \circ \mathcal{R}_{\Pi_A})$ is given by*

$$\mathbb{P} \mapsto \tilde{\mathbb{P}} = (\mathbb{B}\ \mathbb{A})\ \mathbb{P}\ (\mathbb{A}\ \mathbb{B}) = (-\ \mathbb{B}\ \mathbb{A})\ \mathbb{P}\ (-\ \overline{\mathbb{B}\ \mathbb{A}})$$

Let $\mathcal{R}_{\Pi_A} = \mathcal{R}_{\boldsymbol{V}}^{\theta}, \quad \mathcal{R}_{\Pi_B} = \mathcal{R}_{\boldsymbol{V}}^{\phi}$

$$\mathcal{R}_{\boldsymbol{V}}^{\psi} = (\mathcal{R}_{\Pi_B} \circ \mathcal{R}_{\Pi_A}) \equiv \mathcal{R}_{\boldsymbol{V}}^{\phi} \circ \mathcal{R}_{\boldsymbol{V}}^{\theta} \circ \mathcal{R}_{P}^{\mathrm{Pi}} \circ \mathcal{R}_{\boldsymbol{V}}^{-\phi} \circ \mathcal{R}_{\boldsymbol{V}}^{-\theta}$$

This is equivalent to

$$\tilde{\mathbb{P}} = (\mathbb{B}\,\mathbb{A})\,\mathbb{P}\,(\,\overline{\mathbb{A}}\,\overline{\mathbb{B}})$$

$$\tilde{\mathbb{P}} = (\mathbb{B}\,\mathbb{A})\,\mathbb{P}\,(\overline{\mathbb{B}\mathbb{A}}) \qquad \text{by Ex. 6 (i)}$$

$$= (-\mathbb{B}\,\mathbb{A})\,\mathbb{P}\,(-\overline{\mathbb{B}\mathbb{A}}) \qquad \text{by Ex. 6(iv)} \qquad (2)$$

QED

Example rotations and reflections are plotted in Figure 1. Quaternion multiplication gives precisely the same result as the reflections in Π_A *and* Π_B.

(v) Use (27) to show that $-\mathbb{B}\,\mathbb{A} = \cos(\psi/2) + \mathbf{V}\sin(\psi/2)$*, thereby simultaneously proving (29) and (28).*

$$\mathbb{V}\,\mathbb{W} = -\,\mathbf{V}\cdot\mathbf{W} + \mathbf{V}\times\mathbf{W} \qquad (27)$$

$-\mathbb{B}\,\mathbb{A}$ is the product two pure quaternions, so it has the structure $-(-\mathbf{B}\cdot\mathbf{A} + \mathbf{B}\times\mathbf{A}) = (\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B})$, and since (p. 292) quaternions can be written as rotation matrices and the **general** quaternion rotation matrix can be written $\mathbb{R}_{\mathbf{v}}^{\psi} = \cos(\psi/2) + \mathbf{V}\sin(\psi/2)$, then we can write the quaternion product with (28). Then the real part ($\mathbf{A}\cdot\mathbf{B}$) corresponds to $\cos(\psi/2)$ and the pure quaternion part ($\mathbf{A}\times\mathbf{B}$) corresponds to $\mathbf{V}\sin(\psi/2)$. The fact that $\mathbf{V}$ is the axis of rotation agrees with the fact that ($\mathbf{B}\times\mathbf{A}$) is orthogonal to both $\mathbf{B}$ and $\mathbf{A}$.

$$-\mathbb{B}\,\mathbb{A} = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B} = \cos(\psi/2) + \mathbf{V}\sin(\psi/2)$$

$$-\overline{\mathbb{B}\,\mathbb{A}} = \mathbf{A}\cdot\mathbf{B} - \mathbf{A}\times\mathbf{B} = \cos(\psi/2) - \mathbf{V}\sin(\psi/2)$$

$$= \cos(-\psi/2) + \mathbf{V}\sin(-\psi/2) = \mathbb{R}_{\mathbf{v}}^{-\psi}$$

Then using the results (2) from (iv),

$$\tilde{\mathbb{P}} = (-\mathbb{B}\,\mathbb{A})\,\mathbb{P}\,(-\overline{\mathbb{B}\,\mathbb{A}}) = \mathbb{R}_{\mathbf{v}}^{\psi}\,\mathbb{P}\,\mathbb{R}_{\mathbf{v}}^{-\psi} \qquad (29)$$

This proves (28) because it shows that the trigonometric form of the general rotation $\mathbb{R}_{\mathbf{v}}^{\psi}$ satisfies (2) from (iv). It proves (29) because it makes use of (2) and (28), which have been proven.