Needham, Chapter 6, Exercise 6, p. 238-239.
G. Palmer, September 15, 2016, 5:45 AM, PST

6 (i) The conjugate $\overline{\mathbb{V}}$ *of a quaternion* $\mathbb{V} = v + \boldsymbol{V}$ *is defined to be the conjugate transpose* $\mathbb{V}^*$ *of the corresponding matrix. Show that* $\overline{\mathbb{V}} \equiv \mathbb{V}^* = v - \boldsymbol{V}$, *and deduce that* $\mathbb{V}$ *is a pure quaternion (analogous to a purely imaginary complex number) if and only if* $\overline{\mathbb{V}} = -\mathbb{V}$.

$$\mathbb{V} = \mathrm{v} + \mathbf{V} = \mathrm{v} + v_1 \boldsymbol{I} + v_2 \boldsymbol{J} + v_3 \boldsymbol{K}$$

$$\overline{\mathbb{V}} \equiv \mathbb{V}^* = (\mathrm{v} + v_1 \boldsymbol{I} + v_2 \boldsymbol{J} + v_3 \boldsymbol{K})^*$$

$$=\left(\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} + \begin{pmatrix} 0 & v_1 i \\ v_1 i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -v_2 \\ v_2 & 0 \end{pmatrix} + \begin{pmatrix} v_3 i & 0 \\ 0 & -v_3 i \end{pmatrix}\right)^*$$

$$= \begin{pmatrix} \overline{v + v_3 i} & \overline{-v_2 + v_1 i} \\ \overline{v_2 + v_1 i} & \overline{v - v_3 i} \end{pmatrix}^T$$

$$= \begin{pmatrix} v - v_3 i & v_2 - v_1 i \\ -v_2 - v_1 i & v + v_3 i \end{pmatrix}$$

$$= \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} - \begin{pmatrix} 0 & v_1 i \\ v_1 i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -v_2 \\ v_2 & 0 \end{pmatrix} - \begin{pmatrix} v_3 i & 0 \\ 0 & -v_3 i \end{pmatrix}$$

$$= \mathrm{v} - (v_1\mathbf{I} + v_2\mathbf{J} + v_3\mathbf{K}) = \mathrm{v} - \mathbf{V}$$

If $\overline{\mathbb{V}} = -\mathbb{V}$, then v = 0, but if v = 0, then

$$\mathbb{V} = 0 + v_1 \boldsymbol{I} + v_2 \boldsymbol{J} + v_3 \boldsymbol{K} = \mathbf{V},$$

so $\mathbb{V}$ is a pure quaternion. If $\overline{\mathbb{V}} \neq -\mathbb{V}$, then v ≠ 0, and $\mathbb{V}$ is not a pure quaternion by definition.

(ii) The length $|\mathbb{V}|$ *of* $\mathbb{V}$ *is defined (by analogy with complex numbers) by* $|\mathbb{V}|^2 = \mathbb{V}\overline{\mathbb{V}}$. *Show that* $|\mathbb{V}|^2 = v^2 + |V|^2 = |\overline{\mathbb{V}}|^2$.

$$|\mathbb{V}|^2 = \mathbb{V}\overline{\mathbb{V}} = (\mathrm{v} + v_1 \boldsymbol{I} + v_2 \boldsymbol{J} + v_3 \boldsymbol{K})(\mathrm{v} - v_1 \boldsymbol{I} - v_2 \boldsymbol{J} - v_3 \boldsymbol{K})$$

$$\begin{aligned}
&= v^2 - \mathrm{v}v_1\mathbf{I} - \mathrm{v}v_2 \boldsymbol{J} - \mathrm{v}v_3 \boldsymbol{K} \\
&+ \mathrm{v}v_1 \boldsymbol{I} - (v_1 \boldsymbol{I})^2 - v_1 v_2\mathbf{IJ} - v_1 v_3\mathbf{IK} \\
&+ \mathrm{v}v_2 \boldsymbol{J} - v_1 v_2 \mathbf{JI} - (v_2 \boldsymbol{J})^2 - v_2 v_3 \mathbf{JK} \\
&+ \mathrm{v}v_3\mathbf{K} - v_1 v_3\mathbf{KI} - v_2 v_3\mathbf{KJ} - (v_3 \boldsymbol{K})^2
\end{aligned}$$

$$\begin{aligned}
&= v^2 + v_1^2 + v_2^2 + v_3^2 \\
&+ (-\mathrm{v}v_1\mathbf{I} + \mathrm{v}v_1 \boldsymbol{I}) + (-\mathrm{v}v_2 \boldsymbol{J} + \mathrm{v}v_2 \boldsymbol{J}) \\
&+ (v_1 v_2 \boldsymbol{K} - v_1 v_2\mathbf{K}) + (v_1 v_3\mathbf{J} - v_1 v_3\mathbf{J}) \\
&+ (v_2 v_3\mathbf{I} - v_2 v_3 \boldsymbol{I}) + (\mathrm{v}v_3\mathbf{K} - \mathrm{v}v_3 \boldsymbol{K})
\end{aligned}$$

$$= v^2 + |V|^2 = \mathbb{V}\overline{\mathbb{V}} = -\overline{\mathbb{V}}(-\mathbb{V}) = \overline{\mathbb{V}}\,\mathbb{V} = |\overline{\mathbb{V}}|^2$$

(iii) If $|\mathbb{V}| = 1$, then $\mathbb{V}$ is called a unit quaternion. Verify that $\mathbb{R}_{\boldsymbol{v}}^{\psi}$ [see (28)] is a unit quaternion, and that $\overline{\mathbb{R}_{\boldsymbol{v}}^{\psi}} = [\mathbb{R}_{\boldsymbol{v}}^{\psi}]^* = \mathbb{R}_{\boldsymbol{v}}^{-\psi}$.

Let C = cos ψ/2, S = sin ψ/2.

If $\mathbb{R}_{\boldsymbol{v}}^{\psi}$ is a unit quaternion, then

$$1 \equiv \left| \mathbb{R}_{\boldsymbol{v}}^{\psi} \right|^2 = \mathbb{R}_{\boldsymbol{v}}^{\psi}\overline{\mathbb{R}_{\boldsymbol{v}}^{\psi}} = [\mathbb{R}_{\boldsymbol{v}}^{\psi}][\mathbb{R}_{\boldsymbol{v}}^{\psi}]^* \text{ (from (ii) and (i))}$$

$$= \begin{pmatrix} C + \text{inS} & (-m + \text{il})\,S \\ (m + \text{il})\,S & C - \text{inS} \end{pmatrix} . \begin{pmatrix} C - \text{inS} & (m - \text{il})\,S \\ (-m - \text{il})\,S & C + \text{inS} \end{pmatrix}$$

$$= \begin{pmatrix} (C + \text{inS})\,(C - \text{inS}) + (-m - \text{il})\,S(m + \text{il})\,S & (C + \text{inS})\,(m - \text{il})\,S + (-m + \text{il})\,S(C + \text{inS}) \\ (m + \text{il})\,S(m - \text{il})\,S + (C - \text{inS})\,(-m - \text{il})\,S & (m + \text{il})\,S(m - \text{il})\,S + (C - \text{inS})\,(C + \text{inS}) \end{pmatrix}$$

$$= \begin{pmatrix} \left(C^2 + n^2 S^2\right) + \left(l^2 + m^2\right) S^2 & 0 \\ 0 & \left(m^2 + l^2\right) S^2 + \left(C^2 + n^2 S^2\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(C^2 + \left(l^2 + m^2 + n^2\right) S^2 & 0 \\ 0 & \left(l^2 + m^2 + n^2\right) S^2 + C^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \equiv 1$$

and that $\overline{\mathbb{R}_{\boldsymbol{v}}^{\psi}} = [\mathbb{R}_{\boldsymbol{v}}^{\psi}]^* = \mathbb{R}_{\boldsymbol{v}}^{-\psi}$

$\overline{\mathbb{R}_{\boldsymbol{v}}^{\psi}} = [\mathbb{R}_{\boldsymbol{v}}^{\psi}]^*$ by (i)

$\overline{\mathbb{R}_{\boldsymbol{v}}^{\psi}} = \overline{C + \boldsymbol{V}S}$	by (28))
= C + $\overline{V}$S	because C and S are reals)
= C - **V**S	from (i)
= C + **V**(-S)	because -sin(θ) = sin(-θ)
= cos(ψ/2) + **V**sin(-ψ/2)	
= cos(-ψ/2) + **V**sin(-ψ/2)	cos(-ψ/2) = cos(ψ/2)

$= \mathbb{R}_{\boldsymbol{v}}^{-\psi}$

QED

(iv) Show that $\overline{\mathbb{V}\,\mathbb{W}} = \overline{\mathbb{W}}\ \overline{\mathbb{V}}$ *and deduce that* $|\mathbb{V}\,\mathbb{W}| = |\mathbb{V}|\,|\mathbb{W}|$*. Thus, for example, the product of two unit quaternions is another quaternion.*

$\mathbb{V}\,\mathbb{W} = (vw - \mathbf{V} \bullet \mathbf{W}) + (v\mathbf{W} + w\mathbf{V} + \mathbf{V}x\mathbf{W})$ (26)

$\overline{\mathbb{V}\,\mathbb{W}} = (vw - \mathbf{V} \bullet \mathbf{W}) - (v\mathbf{W} + w\mathbf{V} + \mathbf{V}x\mathbf{W})$ (i), this exercise $\overline{\mathbb{V}} = v - \mathbf{V}$

$= (vw - (-\overline{\boldsymbol{V}}) \bullet (-\overline{\boldsymbol{W}})) - (v(-\overline{\boldsymbol{W}}) + w(-\overline{\boldsymbol{V}}) + (-\overline{\boldsymbol{V}})x(-\overline{\boldsymbol{W}}))$ (i), this exercise, $\overline{\mathbb{V}} = -\mathbb{V}$ if $\mathbb{V}$ is a pure quaternion

$= (vw - \overline{\boldsymbol{V}} \bullet \overline{\boldsymbol{W}}) - (-v\overline{\boldsymbol{W}} - w\overline{\boldsymbol{V}} + \overline{\boldsymbol{V}}x\overline{\boldsymbol{W}})$

$= (vw - \overline{\boldsymbol{V}} \bullet \overline{\boldsymbol{W}}) - (-v\overline{\boldsymbol{W}} - w\overline{\boldsymbol{V}} - \overline{\boldsymbol{W}}x\overline{\boldsymbol{V}})$

$= (wv - \overline{\boldsymbol{W}} \bullet \overline{\boldsymbol{V}}) + (w\overline{\boldsymbol{V}} + v\overline{\boldsymbol{W}} + \overline{\boldsymbol{W}}x\overline{\boldsymbol{V}})$

$= \overline{\mathbb{W}}\ \overline{\mathbb{V}}$

deduce that $|\mathbb{V}\,\mathbb{W}| = |\mathbb{V}|\,|\mathbb{W}|$

$|\mathbb{V}\,\mathbb{W}|^2 = \mathbb{V}\,\mathbb{W}\,\overline{\mathbb{V}\,\mathbb{W}}$

$= \mathbb{V}\,\mathbb{W}\,\overline{\mathbb{W}}\ \overline{\mathbb{V}}$ by (iv)

$= \mathbb{V}\,|\mathbb{W}|^2\,\overline{\mathbb{V}}$

$= \mathbb{V}\,\overline{\mathbb{V}}\,|\mathbb{W}|^2$ because the position of the real doesn't affect the result

$= |\mathbb{V}|^2\,|\mathbb{W}|^2$

$|\mathbb{V}\,\mathbb{W}| = |\mathbb{V}|\,|\mathbb{W}|$ take sqrt both sides

QED

(v) Show that $\mathbb{A}$ *is a pure, unit quaternion if and only if* $\mathbb{A}^2 = -1$.

Let $\mathbb{A} \equiv \mathcal{R}_{\boldsymbol{v}}^{\psi} = \cos(\psi/2) + \mathbf{V}\sin(\psi/2)$, where **V** is a pure quaternion. There are indications that **V** is also a unit quaternion, but we will not assume that it is. See p. 289, where $l^2 + m^2 + n^2 = 1$, and p. 291, where $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, which becomes the **v** in $\mathbb{R}_{\boldsymbol{v}}^{\psi} = \cos(\psi/2) + \mathbf{V}\sin(\psi/2)$ on p. 292, where $\mathbf{V} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

If $\mathbb{A}^2 = -1$, then

$$(\cos(\psi/2) + \mathbf{V}\sin(\psi/2))^2 = -1$$

$$C^2 + 2\,\mathbf{V}\,\mathrm{CS} + \boldsymbol{V}^2\,S^2 = -1 \quad \text{(using the notation introduced in (iii))}$$

$\mathbb{A}$ can only be a pure quaternion if it is a binary rotation (so the $\cos(\psi/2)$ term goes to zero), which means that $\psi = \pi$. Then

$$0^2 + 2\,\mathbf{V}\,0\,1 + \boldsymbol{V}^2 = -1$$

$\boldsymbol{V}^2 = -1$, so $\mathbb{A}^2$ must equal $\boldsymbol{V}^2$, so $\mathbb{A}$ must be a pure quaternion.

Since $\mathbb{A}^2 = -1$,

$$-\mathbb{A}\,\mathbb{A} = 1$$

$$-\mathbb{A} = \overline{\mathbb{A}} \Longrightarrow \overline{\mathbb{A}}\,\mathbb{A} = 1$$

$$|\overline{\mathbb{A}}\,|^2 = \;|\;\mathbb{A}\;|^2 = 1$$

This implies that $|\mathbb{A}| = 1$, so A is also a unit quaternion.

If $\mathbb{A}^2 \mp -1$, then $\mathbb{A}$ can not be a unit quaternion and $(\cos(\psi/2) + \mathbf{V}\sin(\psi/2))^2 \mp -1$. Since V is a unit quaternion, $\cos(\psi/2) \mp 0$, so $\mathbb{A}$ is can not be a pure quaternion. Therefore $\mathbb{A}$ is a pure, unit quaternion if and only if $\mathbb{A}^2 = -1$.

QED

*(vi) Show that any quaternion $\mathbb{Q}$ can be expressed as $\mathbb{Q} = |\mathbb{Q}|\,\mathbb{R}_{\boldsymbol{v}}^{\psi}$ for some **v** and some ψ.*

$$\mathbb{R}_{\boldsymbol{v}}^{\psi} = \cos(\psi/2) + \mathbf{V}\sin(\psi/2) \qquad (28),\ \text{p. } 292$$

$$\mathbb{R}_{\boldsymbol{v}}^{\psi} \equiv \begin{pmatrix} \cos(\psi/2) + \mathrm{i}n\sin(\psi/2) & (-m + \mathrm{i}l)\sin(\psi/2) \\ (m + \mathrm{i}l)\sin(\psi/2) & \cos(\psi/2) - \mathrm{i}n\sin(\psi/2) \end{pmatrix} = [R_{\boldsymbol{v}}^{\psi}] \qquad (21),\ \text{p. } 289$$

We have shown in our notes (nh.ch6.notes.questions.SII.pdf, p. 11) that (28) is equivalent to (21), as Needham states on p. 292, paragraph 1, where $[\mathbb{R}_{\boldsymbol{v}}^{\psi}]$ is a general, normalized rotation matrix for some unit vector **v** and some ψ, so $|\mathbb{R}_{\boldsymbol{v}}^{\psi}| = 1$. It follows that we can write $\mathbb{Q} = |\mathbb{Q}|\,\mathbb{R}_{\boldsymbol{v}}^{\psi}$ for any quaternion $\mathbb{Q}$ for some unit vector **v** and some ψ.

(vii) Suppose we generalize the transformation (29) to $\mathbb{P} \to \tilde{\mathbb{P}} = \mathbb{Q}\mathbb{P}\overline{\mathbb{Q}}$, where $\mathbb{Q}$ is an arbitrary quaternion. When interpreted in this way, deduce that $\mathbb{Q}$ represents a dilative rotation of space, and the

product of two quaternions represents the composition of the corresponding dilative rotations. [This confirms the claim at the end of Chapter 1.]

Needham states that quaternions are centred at *O* (p. 44, paragraph 2). In (vi) we showed that we can represent $\mathbb{Q}$ as $\mathbb{Q} = |\mathbb{Q}|\, \mathbb{R}_{\boldsymbol{v}}^{\psi}$, which is a rotation of a point in space about an axis v multiplied by an expansion $|\mathbb{Q}|$ from *O*, so it is a dilative rotation of space.

I don't understand what we are supposed to infer from $\tilde{\mathbb{P}} = \mathbb{Q}\mathbb{P}\overline{\mathbb{Q}}$, but we can examine the product composition of two quaternions each written as an expansion times a rotation ψ or θ about unit vectors v or w:

$$\mathbb{Q}\ \mathbb{S} = |\mathbb{Q}|\, \mathbb{R}_{\boldsymbol{v}}^{\psi}\ |\mathbb{S}|\ \mathbb{R}_{w}^{\theta} = |\mathbb{Q}|\ |\mathbb{S}|\ \mathbb{R}_{\boldsymbol{v}}^{\psi}\ \mathbb{R}_{w}^{\theta} = |\mathbb{Q}\ \mathbb{S}|\ \mathbb{R}_{\boldsymbol{v}}^{\psi}\ \mathbb{R}_{w}^{\theta}$$

Since the product of two rotations is a rotation (because it is the stereographic projection of a composition of two Möbius transformations), we can write this as

$|\mathbb{Q}\ \mathbb{S}|\ \mathbb{R}_{\boldsymbol{u}}^{\phi}$, which is a dilative rotation of space

where $\mathbb{R}_{\boldsymbol{u}}^{\phi} \equiv [\mathbb{R}_{\boldsymbol{u}}^{\phi}] = [\mathbb{R}_{\boldsymbol{v}}^{\psi}]\left[\mathbb{R}_{w}^{\theta}\right]$, u is a vector, and ϕ is a rotation about u. A solution for ϕ and u might be found in the format of (21).