Needham, Chapter 6, Exercise 5, p. 328.
G. Palmer, August 17, 2016

5 Show that the Möbius transformations (20) do indeed satisfy the differential equation (17).

(20) $R_a^{\psi}(z) = \frac{Az+B}{-\overline{B}z+\overline{A}}$

(17) $f'(z) = \frac{1+|f(z)|^2}{1+|z|^2}$

Vasco has shown that this should be $|f'(z)|$.

We begin by differentiating $R_a^{\psi}(z)$.

$$f'(z) = dR_a^{\psi}(z)/dz = \left(\frac{Az+B}{-\overline{B}z+\overline{A}}\right)'$$

$$= \frac{(-\overline{B}z+\overline{A})A-(Az+B)(-\overline{B})}{(-\overline{B}z+\overline{A})^2}, \text{ by the quotient rule}$$

$$= \frac{(-A\overline{B}z+A\overline{A}+A\overline{B}z+B\overline{B})}{(-\overline{B}z+\overline{A})^2}$$

$$= \frac{A\overline{A}+B\overline{B}}{(-\overline{B}z+\overline{A})^2}$$

$$= \frac{|A|^2+|B|^2}{(-\overline{B}z+\overline{A})^2}$$

$$|f'(z)| = \frac{|A|^2+|B|^2}{\left|(-\overline{B}z+\overline{A})^2\right|} = \frac{|A|^2+|B|^2}{|-\overline{B}z+\overline{A}|^2}$$

Instantiate f(z) in (17) with $\frac{Az+B}{-\overline{B}z+\overline{A}}$

$$1 + |f|^2 = 1 + \left|\frac{Az+B}{-\overline{B}z+\overline{A}}\right|^2$$

$$= 1 + \frac{(Az+B)(\overline{Az+B})}{(-\overline{B}z+\overline{A})(\overline{-\overline{B}z+\overline{A}})} = 1 + \frac{(Az+B)(\overline{Az+B})}{(-\overline{B}z+\overline{A})(-B\overline{z}+A)}$$

$$= \frac{(-\overline{B}z+\overline{A})(-B\overline{z}+A)+(Az+B)(\overline{Az+B})}{(-\overline{B}z+\overline{A})(-B\overline{z}+A)}$$

$$= \frac{|A|^2+|B|^2+|A|^2|z|^2+|B|^2|z|^2}{(-\overline{B}z+\overline{A})(-B\overline{z}+A)}$$

$$= \frac{(1+|z|^2)(|A|^2+|B|^2)}{(-\overline{B}z+\overline{A})(-B\overline{z}+A)}$$

$$f'(z) = \frac{1+|f(z)|^2}{1+|z|^2} = \frac{\frac{(1+|z|^2)(|A|^2+|B|^2)}{(-\overline{B}z+\overline{A})(-B\overline{z}+A)}}{1+|z|^2}$$

$$= \frac{|A|^2+|B|^2}{\left|-\overline{B}z+\overline{A}\right|^2} = |f'(z)|$$

It appears that we have shown that $R_a^{\psi}(z)$ satisfies the equation

$$|f'(z)| = \frac{1+|f(z)|^2}{1+|z|^2}$$