Needham, Chapter 6, Exercise 4, p. 328.
G. Palmer, August 29, 2016, 10:20 PM

4 Use (41), p. 166 to show that if the Möbius transformation M(z) has fixed points $\xi_\pm$ and the multiplier associated with ξ_+ is m, then

$$[M] = \begin{pmatrix} 1 & -\xi_+ \\ 1 & -\xi_- \end{pmatrix}^{-1} \begin{pmatrix} \sqrt{m} & 0 \\ 0 & 1\big/\sqrt{m} \end{pmatrix} \begin{pmatrix} 1 & -\xi_+ \\ 1 & -\xi_- \end{pmatrix}$$

[Parentheses for matrices is *Mathematica* default style, which I am unable to defeat.]

By putting $\xi_+ = a$, $m = e^{-i\psi}$, and $\xi_- = -(1/\overline{a})$, deduce (19), p. 288. [Hint: Remember that you are free to multiply a Möbius matrix by constant.]

This exercise is based upon Figure [29], p. 163.

$$\frac{w-\xi_+}{w-\xi_-} = m\left(\frac{z-\xi_+}{z-\xi_-}\right) \qquad \text{(41), p. 166}$$

$F(w) = m\, F(z)$ from equation p. 163 and figure [29]

$\tilde{M}(\tilde{z}) = m\, \tilde{z}$ p. 164, $\tilde{z} = F(z)$

On p. 166, it is given that

$$F \circ M = \tilde{M} \circ F$$

On the left of [29], we have w = M(z). Therefore we can write

$$F(M(z)) = F(w) = m\, F(z)$$

Taking just $F(w) = m\, F(z)$, we have

$$\frac{w-\xi_+}{w-\xi_-} = m\left(\frac{z-\xi_+}{z-\xi_-}\right)$$

Rearrange the compositions:

$$F \circ M = \tilde{M} \circ F$$

$$M = F^{-1} \circ F \circ M = F^{-1} \circ \tilde{M} \circ F$$

$[M] = [F]^{-1}\, [\tilde{M}]\, [F]$ rewrite as matrix multiplication

$$= [F]^{-1}\,[\mathrm{m}\;\tilde{z}]\,[\mathrm{F}] = \;[F]^{-1}[\tfrac{\mathrm{m}\tilde{z}}{1}]\,[\mathrm{F}] = [F]^{-1}\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}[\mathrm{F}]$$

$$[\mathrm{M}] = \begin{pmatrix} 1 & -\xi_+ \\ 1 & -\xi_- \end{pmatrix}^{-1}\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\xi_+ \\ 1 & -\xi_- \end{pmatrix}$$

$= \begin{pmatrix} 1 & -\xi_+ \\ 1 & -\xi_- \end{pmatrix}^{-1}\begin{pmatrix} \sqrt{m} & 0 \\ 0 & \frac{1}{\sqrt{m}} \end{pmatrix}\begin{pmatrix} 1 & -\xi_+ \\ 1 & -\xi_- \end{pmatrix}$, multiplying middle matrix $\tilde{M}$ by $\frac{1}{\sqrt{\mathrm{ad-bc}}} = \frac{1}{\sqrt{m}}$ to normalize.

Vasco points out that in this case normalizing is just multiplying by a constant, which is what are allowed to do anyway. Yet it seems more than a coincidence that the exponentials in the final answer reflect the normalized form.

This is what we wanted to show. *By putting $\xi_+ = a$, $m = e^{-i\psi}$, and $\xi_- = -(1/\overline{a})$, deduce (19), p. 288.* There appears to be no need to normalize F^{-1} or F (p. 163, Note at bottom).

$$= \frac{1}{\det(F)}\begin{pmatrix} -\xi_- & \xi_+ \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{m} & -\xi_+\sqrt{m} \\ \frac{1}{\sqrt{m}} & -\xi_-\frac{1}{\sqrt{m}} \end{pmatrix}$$

$$= \frac{1}{\xi_+ - \xi_-}\begin{pmatrix} -\xi_-\sqrt{\mathrm{m}} + \xi_+\frac{1}{\sqrt{\mathrm{m}}} & \xi_+\xi_-\left(\sqrt{\mathrm{m}} - \frac{1}{\sqrt{\mathrm{m}}}\right) \\ -\sqrt{\mathrm{m}} + \frac{1}{\sqrt{\mathrm{m}}} & \xi_+\sqrt{\mathrm{m}} - \xi_-\frac{1}{\sqrt{\mathrm{m}}} \end{pmatrix}$$

$$= \frac{1}{a + 1/\overline{a}}\begin{pmatrix} (1/\overline{a})\,e^{-\mathrm{i}\psi/2} + \mathrm{a}e^{\mathrm{i}\psi/2} & -a\,(1/\overline{a})\left(e^{-\mathrm{i}\psi/2} - e^{\mathrm{i}\psi/2}\right) \\ -e^{-\mathrm{i}\psi/2} + e^{\mathrm{i}\psi/2} & \mathrm{a}e^{-\mathrm{i}\psi/2} + (1/\overline{a})\,e^{\mathrm{i}\psi/2} \end{pmatrix}$$

$$= \frac{\overline{a}}{|a|^2 + 1}\begin{pmatrix} \mathrm{a}e^{\mathrm{i}\psi/2} + (1/\overline{a})\,e^{-\mathrm{i}\psi/2} & \frac{a}{a}\,2\,i\sin(\psi/2) \\ 2\,i\sin(\psi/2) & \mathrm{a}e^{-\mathrm{i}\psi/2} + (1/\overline{a})\,e^{\mathrm{i}\psi/2} \end{pmatrix}$$

$\equiv \overline{a}\begin{pmatrix} \mathrm{a}e^{\mathrm{i}\psi/2} + (1/\overline{a})\,e^{-\mathrm{i}\psi/2} & \frac{a}{a}\,2\,i\sin(\psi/2) \\ 2\,i\sin(\psi/2) & \mathrm{a}e^{-\mathrm{i}\psi/2} + (1/\overline{a})\,e^{\mathrm{i}\psi/2} \end{pmatrix}$, by the hint, multiply by $|\,a\,|^2 + 1$

$$= \begin{pmatrix} e^{\mathrm{i}\psi/2}\,|\,a\,|^2 + e^{-\mathrm{i}\psi/2} & 2\,i\,a\sin(\psi/2) \\ 2\,i\,\overline{a}\sin(\psi/2) & e^{-\mathrm{i}\psi/2}\,|\,a\,|^2 + e^{\mathrm{i}\psi/2} \end{pmatrix}$$

QED