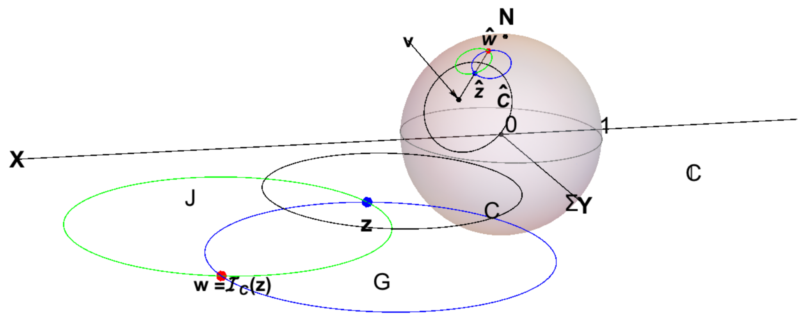
Needham, Chapter 6, Exercise 3, p. 328
G. Palmer, October 8, 2016, 3:15 pm PST

Fig 1. Projection from vertex of cone on $\hat{C}$ induces reflection in C



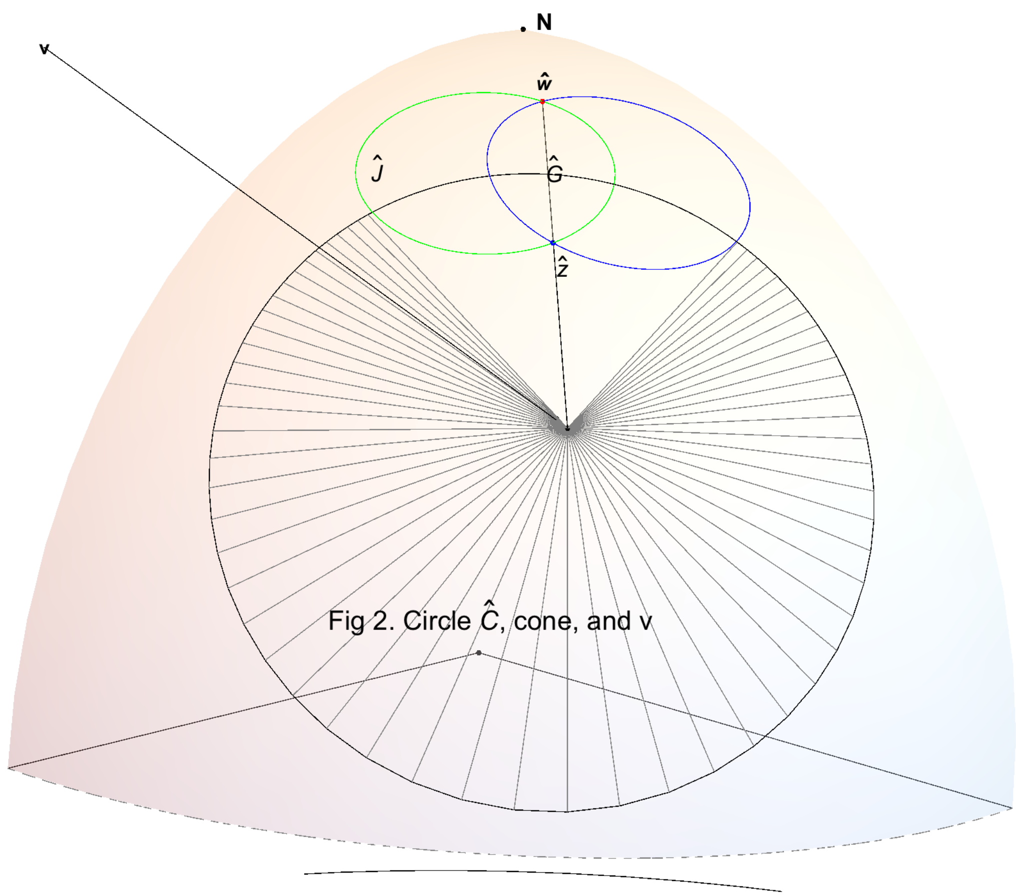


Fig 2. Circle $\hat{C}$, cone, and v

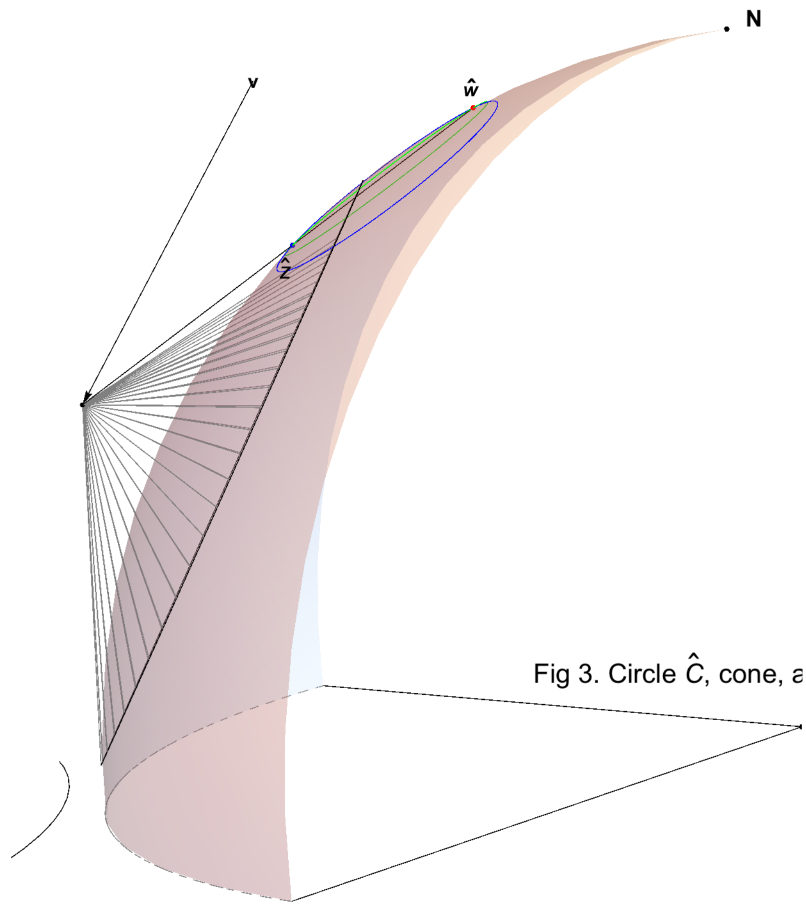


Fig 3. Circle $\hat{C}$, cone, a

3 Let C be a circle in $\mathbb{C}$, and let $\hat{C}$ be it's stereographic image on Σ. If $\hat{C}$ is a great circle, then (18) says that $\mathcal{I}_C$ stereographically induces reflection of Σ in $\hat{C}$, but what transformation is induced if C is an arbitrary circle? Generalize the argument of figure [14] to show that $\mathcal{I}_C$ becomes [the] projection from the vertex v of the cone that touches Σ along $\hat{C}$. That is, if $w = \mathcal{I}_C(z)$, then $\hat{w}$ is the second intersection point of Σ with the line in space that passes through the vertex v and *the point $\hat{z}$. Explain how (18) may be viewed as a limiting case of this more general result.*

Following Vasco, we have changed the problem to read:

Let C be a circle in $\mathbb{C}$, and let $\hat{C}$ be it's stereographic image on Σ. If $\hat{C}$ is a great circle, then (18) says *that $\mathcal{I}_C$ stereographically induces reflection of Σ in $\hat{C}$, but what transformation is induced if $\hat{C}$ is an arbitrary circle? Generalize the argument of figure [14] to show that $\mathcal{I}_C$ becomes [the]* projection from the vertex v of the cone that touches Σ along $\hat{C}$. That is, if w = $\mathcal{I}_C$(z), then $\hat{w}$ is the second intersection point of Σ with the line in space that passes through* the vertex v and *the point $\hat{z}$. Explain how (18) may be viewed as a limiting case of this more general result.*

Notation and interpretation*:* Let z, w, C, G, and J be points or circles in ℂ. Let $\hat{z}$, $\hat{w}$, $\hat{C}$, $\hat{G}$, and $\hat{J}$ be points or circles in Σ. The "^" symbol does not in itself symbolize stereographic projection. In sentence three, "projection from the vertex v" is not stereographic projection. It is just extension of a line from v until it intersects Σ at two points, $\hat{z}$ and P($\hat{z}$). The cone that "touches Σ along $\hat{C}$" can have only one vertex because it is tangent to Σ at $\hat{C}$. A line from v to ($\hat{z}$ not in Σ) is not actually a line in the cone; rather it is a line inside the space within the cone.

a) Draw an arbitrary circle $\hat{C}$ in Σ.
b) Draw a point v at the vertex of a regular cone based in $\hat{C}$. All straight lines from v to $\hat{C}$ are tangents to Σ and orthogonal to $\hat{C}$.
c) Plot a point $\hat{z}$ inside $\hat{C}$.
d) Draw a line from v through $\hat{z}$, where it intersects Σ and extend (project) it until it intersects Σ a second time. Label the second point of intersection P($\hat{z}$) for non-stereographic projection.
e) Use the two points of intersection to draw two circles $\hat{G}$ and $\hat{J}$ orthogonal to $\hat{C}$ in Σ. Vasco has reasoned in the Forum that they are orthogonal to $\hat{C}$ because a line drawn from v to a point of intersection of either $\hat{G}$ or $\hat{J}$ with $\hat{C}$ touches Σ at the point and is orthogonal to $\hat{C}$, so it also touches $\hat{G}$ or $\hat{J}$ at the same point and is therefore tangent to one of them. Hence, $\hat{G}$ or $\hat{J}$ must be orthogonal to $\hat{C}$. Vasco: "The crucial point is that the circle [$\hat{G}$ or $\hat{J}$] and the line [from v to a point of intersection of either $\hat{G}$ or $\hat{J}$ with $\hat{C}$] are in the same plane - that defined by v and the two points of intersection of the circle with $\hat{C}$, and so if the circle touches the line then that is the definition of a tangent!". [material in brackets added]
f) Apply stereographic inversion to $\hat{C}$, $\hat{G}$, $\hat{J}$, $\hat{z}$, and P($\hat{z}$) to obtain circles C, G, J, and points z and P(z) in ℂ. G and J must be orthogonal to C.
g) The two points of intersection $\hat{z}$, and P($\hat{z}$) of $\hat{G}$ and $\hat{J}$ are carried to points of intersection z and P(z) of G and J in ℂ, where we know they represent mutual inversions in C (the projection of $\hat{C}$). Thus, $\mathcal{I}_C$ *becomes [the]* projection from the vertex v of the cone that touches Σ along* $\hat{C}$.
h) P(z) is the inversion w of z, so P($\hat{z}$) = $\hat{w}$, the second point of intersection in Σ. We have shown that *if w = $\mathcal{I}_C$(z), then $\hat{w}$ is the second intersection point of Σ with the line in space that passes through* the vertex v and *the point $\hat{z}$.*

Explain how (18) may be viewed as a limiting case of this more general result.

According to (18), *reflection of Σ in a line* [i.e. great circle] *induces reflection (inversion) of ℂ in the stereographic image of that line.* The great circle in (18) represents the limiting case of $\hat{C}$ in which the point v is moved to an infinite distance from 0. As it does so, the lines from v touching $\hat{C}$ become orthogonal to rays from the origin to Σ, and $\hat{C}$ becomes a great circle.

This seems to raise a point for discussion. Since $\hat{w}$ and $\hat{z}$ are projections of w and z, and since w and z are inversions in C, doesn't this imply that $\hat{w}$ and $\hat{z}$ are inversions in $\hat{C}$ on Σ? If so, then we should be able to say that reflection of Σ in an arbitrary circle induces reflection (inversion) of ℂ in the stereographic image of that circle. This constitutes a generalization of (18).

The plot: It is much easier to create custom objects and apply transformations in ℂ than it is in Σ, where finding points often requires solving vector equations. The circles and points of intersection were created first in ℂ and then projected to Σ by stereographic projection. Then a line was plotted through $\hat{z}$ and $\hat{w}$. To find the vertex, z and w were rotated around the center of C in ℂ and projected to Σ as $\hat{\mathcal{R}}$(z) and $\hat{\mathcal{R}}$(w). Another line was plotted though these two points and extended to meet the first line at v. Neither of these lines are in the cone mentioned in the problem, but the cone would be plotted on this vertex v and the base $\hat{C}$.

*The article "the" is added by me for American readers.