Needham, Chapter 6, Exercise 27, p. 337.
G. Palmer, February 17, 2017, 8:00 pm PST

27 Generalize the transformation from the upper half - plane to the Poincaré disc to obtain a model of hyperbolic space in the interior of the unit sphere. Describe the appearance of h-lines, h-planes, h-spheres, and horospheres in this model, and explain why an h-sphere is intrinsically the same as a Euclidean sphere of different radius.

The question implies that we are to create a 3-dimensional model of hyperbolic space within a particular 3-dimensional object, the unit sphere. Since the *PD* is 2-dimensional, we need a mechanism to project it to three dimensions. This could possibly be done via stereographic projection, but Vasco has described a better approach in his answer. I develop what I think is a version of his model.

The PD can be placed so that the horizon coincides with the equator of the unit sphere. From that position it can be rotated to any orientation which we could write as (θ, ϕ), where θ is the usual angle of rotation about the origin from the x axis and ϕ is an angle of rotation **about the origin** from the z axis. Then, any particular PD could be written as $PD_{(\theta,\phi).}$ A permutation of graded values of θ from 0 to 2Pi and ϕ from -Pi to Pi would fill the volume of the sphere and create a spherical horizon that is infinitely distant from any point inside the sphere, just as the horizon of the PD is infinitely distant from any point in the PD.

To verify this, the metric in any one of these $PD_{(\theta,\phi)}$ (any great circle slice of the sphere) is $d\hat{s} = \Lambda(z)ds = \frac{1}{1-|z|}ds$. $\Lambda(z)$ depends only on the location of z. $|z| < 1$ because we are concerned only with the interior of the sphere. As $|z| \longrightarrow 1$, $d\hat{s} \longrightarrow \infty$. This metric applies only to points in one of the $PD_{(\theta,\phi).}$ But with an infinite number of PDs filling the volume of the sphere, every two points belong to at least one $PD_{(\theta,\phi)}$, so the metric actually applies to any two points in the interior of the sphere with $d\hat{s}$ being a distance in some $PD_{(\theta,\phi)}$. We can see that this is true, because a plane can be plotted containing the center of the sphere and any two points in the sphere. Hence, if $r = |z|$, z on some $PD_{(\theta,\phi)}$, r can be interpreted as an absolute distance specifiable in three dimensions as $\sqrt{X^2 + Y^2 + Z^2}$.

h-lines

In this model, a circular h-line in a $PD_{(\theta,\phi)}$ would appear as the segment of a circle in a plane orthogonal to the sphere, because all $PD_{(\theta,\phi)}$ are orthogonal to the sphere. If the h-line were a diameter in a $PD_{(\theta,\phi)}$, it would appear as a diameter of the sphere in that same $PD_{(\theta,\phi)}$.

h-planes

The $PD_{(\theta,\phi)}$ are h-planes because they are subject to the same metric. On p. 325, we learn that the upper half sphere is also an h-plane which transforms to the UHP under inversion at the point q opposite to the intersection of the UHP and the UHS. If the UHP is an h-plane, then so is any other half-sphere under the right construction.

h-spheres

The metric of the unit h-sphere is $d\hat{s} = \frac{1}{1-|z|}ds$, which can be written as $d\hat{s} = \frac{1}{1-r}ds$. The rewrite suggests that the volume of the h-sphere is filled with hypothetical concentric h-spheres whose surfaces are h-planes. But it should be possible to obtain an h-sphere by using any h-circle as a generator. Pick an h-circle in any $PD_{(\theta,\phi)}$ and find the perpendicular bisector of the two endpoints orthogonal to the sphere. Rotate the h-circle about the perpendicular bisector. The result is a segment or patch of an h-sphere.

But the fact that not all circles in the PD are h-circles suggests that there can be spheres in h-space that are not generated by h-circles and in fact are not h-spheres. We could create such spheres by picking any diameter in a circle and rotating the circle about that diameter. Since these spheres exist in h-space, they are subject to the h-metric. The last sentence of the problem suggests that Needham regards all interior spheres as h-spheres.

horospheres

Find the diameter of any horocycle in any $PD_{(\theta,\phi)}$ where it touches the sphere. This is the diameter orthogonal to the horocycle and the sphere. Rotate the horocycle about this diameter to obtain a horosphere.

Explain why an h-sphere is intrinsically the same as a Euclidean sphere of different radius.

The question hints that any sphere within the unit sphere is an h-sphere, contrary to my formulation of the h-sphere above. I accept the hint.

An h-sphere must be h-centred at the intersection of two orthogonal h-lines in any $PD_{(\theta,\phi)}$. Therefore, like the circle in the PD from which it was generated, an h-sphere must also have an Euclidean centre and radius that are different from the h-centre and h-radius according to the equations given on p. 309.