Needham, Chapter 6, Exercise 26, p. 337.
G. Palmer, February 17, 2017, 10:25 am PST

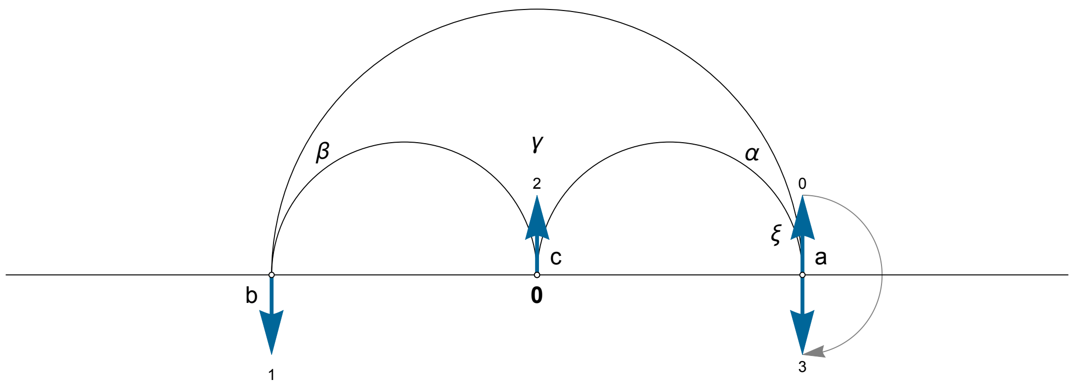

Figure 1. E(Δ) of asymptotic hyperbolic triangle abc

26 Sketch a hyperbolic triangle Δ with vertices (in counterclockwise order) a, b, and c. Let ξ be an infinitesimal vector emanating from a and pointing along the edge ab. Carry ξ to b by h-translating it along this edge. Now carry it to c along bc, and finally carry it home to a, along ca. In Euclidean geometry these three translations would simply cancel, and ξ would return home unaltered. Use your sketch to show that in hyperbolic geometry the composition of these three h-translations is an h-rotation about vertex a through angle E(Δ).

Figure 1 joins the points a, b, and c with h-line segments. The triangle is a very simple construction using three h-lines that meet asymptotically. In Figure 1, the teal colored arrows represent the infinitesimal vector ξ at each point. The small numbers 0 to 3 at the arrow tips indicate their order in the sequence of h-translations. Since every angle is 0, the area $\mathcal{A} = \pi - \alpha - \beta - \gamma = \pi = -E$, so $E = -\pi$. An h-translation does not alter the angle of the infinitesimal vector with the h-line on which it is translated. Consequently, translating a vector along the full length of the h-line rotates it by π or $-\pi$. The rotations generated by the three translations are as follows:

1. π
2. $-\pi$
3. $-\pi$

The sum of the three rotations equals $-\pi = E$. This shows that in hyperbolic geometry the composition of these three h-translations is an h-rotation about vertex a through angle E(Δ).

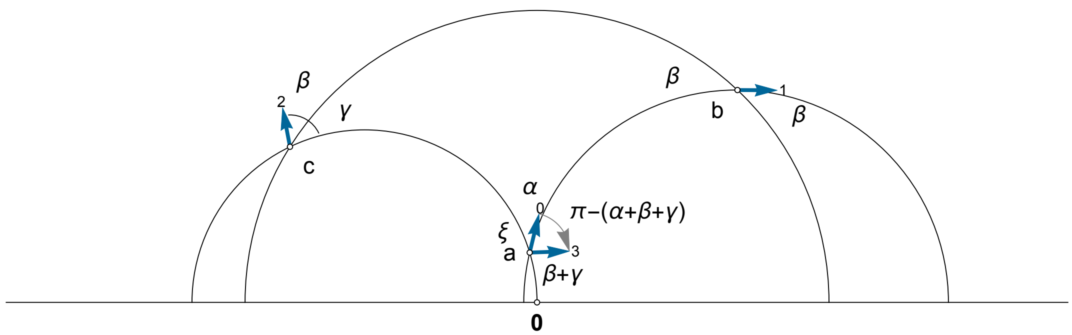

Figure 2. E(Δ) of general hyperbolic triangle abc

The preceding provides an example that illustrates the principle. Figure 2 illustrates a general demonstration. The apparent rotations of ξ arise entirely from the translations, which preserve the angle between ξ and the segment of h-line on which it is translating. The end result is an h-rotation of π-(α+β+γ) about a. In this case, we can see that if α, β, and γ are all treated as positive acute angles, then π-(α+β+γ) is a positive number, but we can see from Figure 2 that the rotation is negative. Since π-(α+β+γ) = -E(Δ), it must be the case that E(Δ) is negative. Therefore, we have shown that *the composition of these three h-translations is an h-rotation about the vertex a through angle E(Δ)*.

To understand this, the (outside) angle β at b can be viewed as an h-rotation of ξ from bc to ab about b. The angle β+γ at c can be viewed as an h-rotation from ca to bc of ξ about c. When ξ is translated home to α, we see that the accumulated result of the three translations is an h-rotation of π-(α+β+γ) about a. This same angle could also be seen as an h-rotation of -(α+β+γ) from ca about a had we not started with ξ aligned with the right hand h-line at a.

[Suppose that Δ is instead a geodesic triangle on an arbitrary surface S of variable curvature. If an inhabitant of S wants to translate ξ along a “straight line” (a geodesic), all he has to do is keep its length constant, and keep the angle it makes with the line constant. This is called parallel transport in differential geometry. The above argument still applies, and so when ξ is parallel transported round Δ, it returns home rotated through E(Δ). By virtue of (6), this angle of rotation is the total amount of curvature inside Δ.]