Needham, Chapter 6, Exercise 25, p. 337.
G. Palmer, May 12, 2020, 4:10 pm PST

Out[1212]=

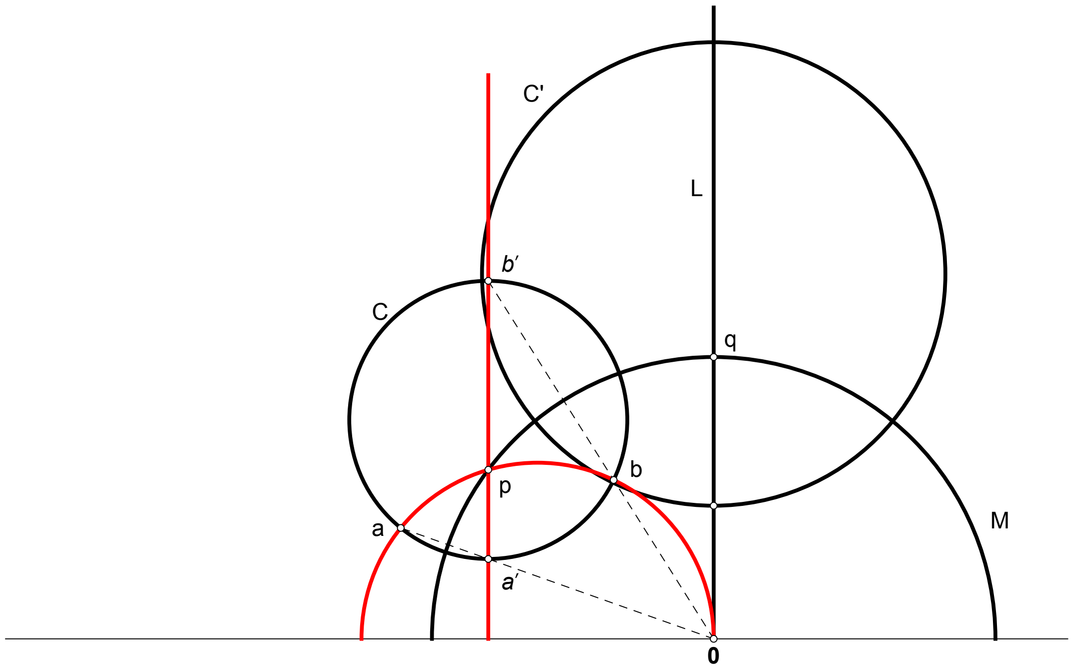

Figure 1. Construction of asymptotic lines in Poincaré UHP

25 Given a point p not on the h-line L, draw an h-circle C of h-radius ρ centred at p. Draw the h-line [M] orthogonal to L through p, cutting L in q. Draw an h-circle C′ of h-radius ρ centred at q, and let q′ be one of the intersection points with L. Through q′ draw the h-line orthogonal to L, cutting C at a and b. Show that the h-lines joining p to a and b are the two asymptotic lines! [Hint: Take L to be a vertical line in the upper half-plane.] What happens if we perform this construction in the Euclidean plane?

Due to the hint, the it seems best to draw two h-lines asymptotic to L. Figure [25] on p. 305 provides a template. The Euclidean centers of C and C' can be found using the equations presented on p. 309, indented italics. The circle through q'orthogonal to L does not cut C, so the question can not be answered as written in the original. The problem seems coherent if sentence 3 is changed from "cutting C at a and b" to "cutting the horizon at a and b" or if it is changed from "draw the h-line orthogonal to L" to "draw the h-line symptotic to L". My first choice was to draw the h-line orthogonal to L, cutting the horizon at a and b. After going through telemeter's document (Forum), I have decided that his solution with asymptotes to L is probably the one intended.

Rewrite of the problem:

Given a point p not on the h-line L, draw an h-circle C of h-radius ρ centred at p. Draw the h-line M orthogonal to L through p, cutting L in q. Draw an h-circle C′ of h-radius ρ centred at q. Draw the h-line asymptotic to L, cutting C at a and b. Show that the h-lines joining p to a and b and joining p to a′ and *b′ are the two asymptotic lines! [Hint: Take L to be a vertical line in the upper half-plane.] What happens if we perform this construction in the Euclidean plane?*

The asymptotes to L are shown in red in Figure 1. In order to find the circular h-line that passes through p and is asymptotic to L, place L at 0 on the horizon and find the point on the horizon that is equidistant from p and 0. In order to find the reflections, use $\mathcal{I}(z) = R^2 \big/ (\bar{z} - \bar{q}) + q$, where R is the Euclidian radius and q is the Euclidean center of the half-line or circle of reflection. In this construction R = |p| = |q|. The fact that the inversions a' and $b'$of a and b in M fall on the proposed asymptotic half-line shows that both are the circular h-line and the half-line are the asymptotic lines from p. The circle C' appears to contribute nothing to this argument.

Out[1249]=

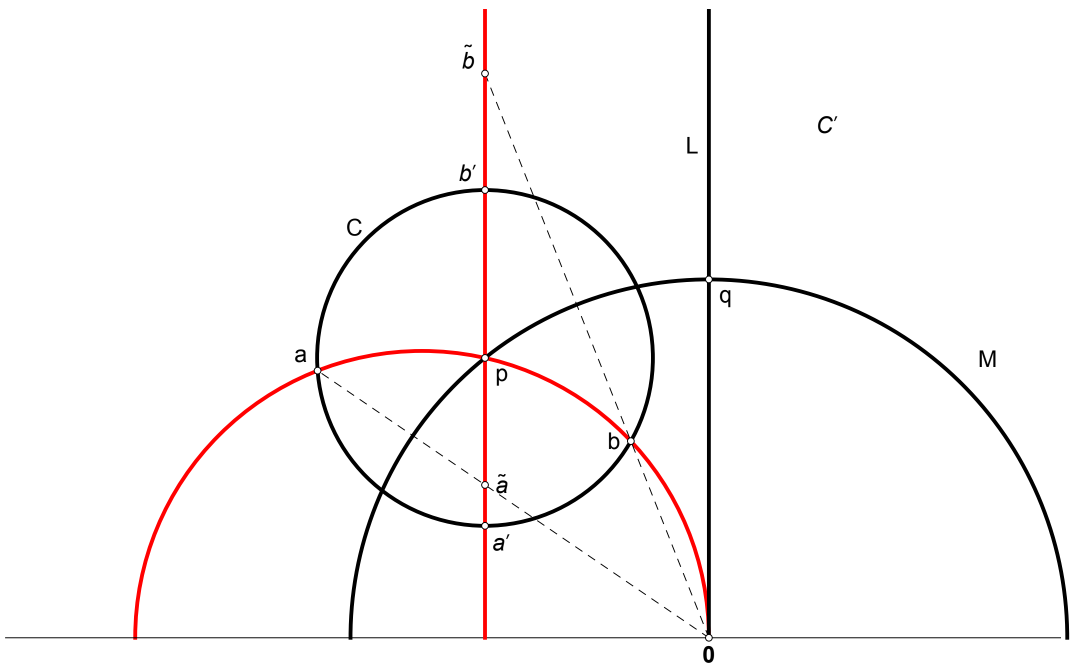

Figure 2. Construction of problem 25 in Euclidean plane

What happens if we perform this construction in the Euclidean plane?

Rewrite the problem accordingly. This involves a bit of guesswork, as "perform this construction" is a vague directive. What are the essential features of the construction in the Euclidean plane that one might perform in the Euclidean plane? Is one to show that a half-circle asymptotic to L and a vertical line at p are reflections about a circle or line? Or is one to calculate the reflections at a and b to show that they miss the mark?

Given a point p not on the line L, draw a circle C of radius ρ centred at p. Draw the half-circle M orthogonal to L through p, cutting L in q. Draw circle C' of radius ρ centred at q. Draw a half-circle asymptotic to L, cutting C at a and b. Show that the segments of circles and lines joining p to a and b and joining p to a' and b' are the two asymptotic lines! [Hint: Take L to be a vertical line in the upper half-plane.] What happens if we perform this construction in the Euclidean plane?

As in Figure 1, label the points where the asymptotic half-circle intersects C as a and b. Label the points vertical line through p intersects C as a' and b'. Now one can see that the inversion of a in M, labeled $\tilde{a}$, does not coincide with a' and the inversion of b in M, labeled $\tilde{b}$, does not coincide with b'. This shows that in the Euclidean construction, the vertical line through p can still be shown to be a hyperbolic asymptote of L, because $\tilde{a}$ and $\tilde{b}$ still fall on the vertical line through p. The circle C' appears to play no significant role in this construction, so it is deleted. The points a' and b' also appear to play no role in the demonstration.