Needham, Chapter 6, Exercise 24, p. 337.
G. Palmer, February 11, 2016, 12:00 noon, PST

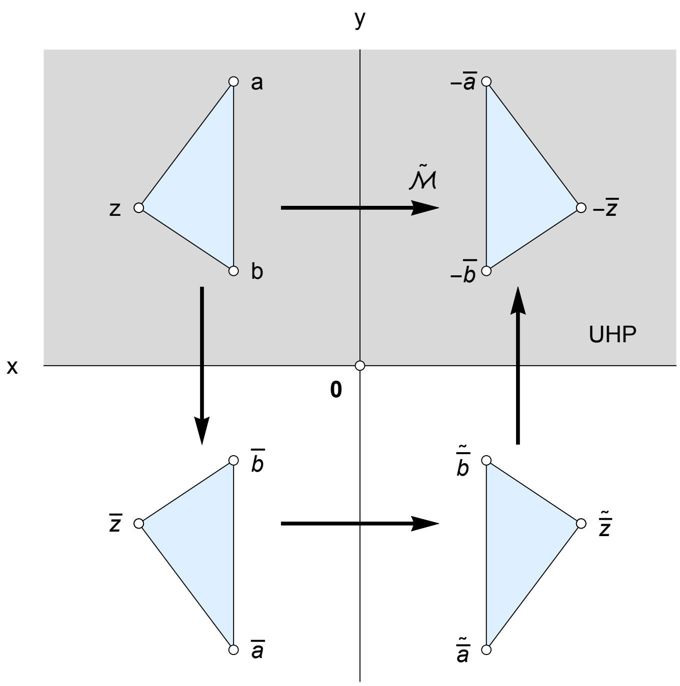

Figure 1. z ↦ f(z) = $-\bar{z}$ is Opposite motion of UHP

24 (i) Use a simple sketch to show that z ↦ f(z) = $-\bar{z}$ is an opposite motion of the Poincaré upper half-plane.

In order to show this opposite motion with three reflections, we have to follow the sequence of reflection by conjugation and reflection across the y axis in the LHP. Then the result will appear as an opposite motion of the UHP (the region above the real line). The upper right panel shows that the angles of $-\bar{z}$ with $-\bar{a}$ and $-\bar{b}$ have opposite sense to the angles of z with a and b in the upper left panel, thus demonstrating an opposite motion (translation plus reflection in a-b line) in the UHP.

(ii) Let $\tilde{\mathcal{M}}$ be an arbitrary opposite motion. By considering (f ∘ $\tilde{\mathcal{M}}$), show that

$$\tilde{\mathcal{M}}(z) = \frac{a\bar{z}+b}{c\bar{z}+d}, \qquad \textit{where a, b, c, d are real, and (ad-bc) > 0}$$

Since f and $\tilde{\mathcal{M}}$ are opposite motions, f ∘ $\tilde{\mathcal{M}}$ must be a direct motion consisting of 2 h-reflections.

$$f \circ \tilde{\mathcal{M}}(z) = -\overline{\tilde{\mathcal{M}}}(z) = -\left(\overline{\frac{a\bar{z}+b}{c\bar{z}+d}}\right) = \frac{-az-b}{cz+d}$$

which is (-1) M(z), where M(z) is the equation for all direct motions of the hyperbolic plane, where where a, b, c, d are real, and (ad-bc) > 0 (p. 313). Since M(z) is normall written $\frac{az+b}{cz+d}$, where (ad-bc) > 0, this suggests that $\tilde{\mathcal{M}}(z)$ should be $\frac{-a\bar{z}-b}{c\bar{z}+d}$. If we want the signs of the coefficients to be the same for $\tilde{\mathcal{M}}(z)$ and M(z), we can write $\alpha = -a$, $\beta = -b$ and

$$\tilde{\mathcal{M}}(z) = \frac{\alpha\bar{z}+\beta}{\gamma\bar{z}+\delta}$$

Since (ad-bc) > 0, $(\alpha\delta-\beta\gamma) = (-ad-(-b)c) = (bc - ad) < 0$.

Now if we wish to replace α, β, δ and γ with a, b, c, and d, as in the question, we have

$$\tilde{\mathcal{M}}(z) = \frac{a\bar{z}+b}{c\bar{z}+d}, \text{ where } (ad-bc) < 0. \qquad (1)$$

So we agree with Vasco that there is probably a mistake in the question.

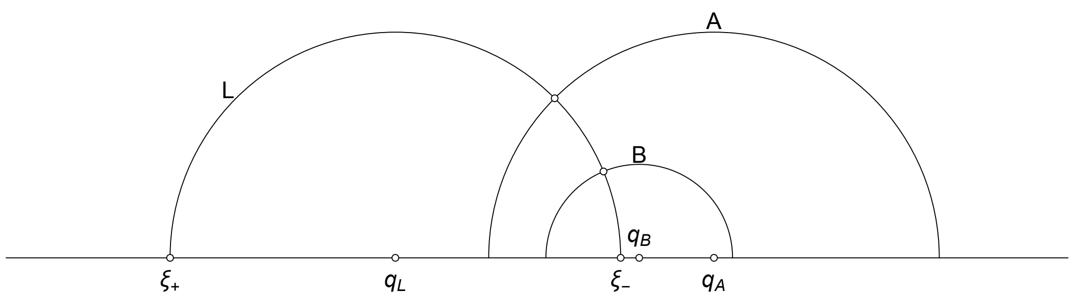

Figure 2. L invariant under $\tilde{\mathcal{M}}$

(iii) Use this formula to show that $\tilde{\mathcal{M}}$ has two fixed points on the horizon (the real axis). If L is the h-line whose ends are these two points, explain why L is invariant under $\tilde{\mathcal{M}}$.

Using equation (1) from (ii), we can find the fixed points by substituting

$z = \tilde{\mathcal{M}}(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ and solving for Im(z)

$$z(c\bar{z} + d) = a\bar{z} + b$$

$$c\,|\,z\,|^2 + dz = a\bar{z} + b$$

$$dz - a\bar{z} = -c\,|\,z\,|^2 + b$$

$\Longrightarrow \mathrm{Im}(dz - a\bar{z}) = 0$, because $(-c\,|\,z\,|^2 + b)$ is real

$$\mathrm{Im}(dz) - \mathrm{Im}(a\bar{z}) = 0$$

$$d\,\mathrm{Im}(z) - a\,\mathrm{Im}(z) = 0$$

$$(d-a)\,\mathrm{Im}(z) = 0 \quad \Longrightarrow \quad \mathrm{Im}(z) = 0,$$

so both the fixed points lie on the real line (horizon) of the UHP.

Since z is real, we substitute x for z in the quadratic equation.

$$cx^2 + dx - ax - b = 0$$

$$cx^2 + (d-a)x - b = 0$$

$$x = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4\,bc}}{2\,c} \qquad (2)$$

Since (ad-bc) < 0, ad < bc and 4bc > 4ad,

$$(d-a)^2 + 4\,bc = a^2 - 2\,ad + d^2 + 4\,bc > a^2 + 2\,ad + d^2 = (a+d)^2 > 0 \quad \text{(subst. 4ad for 4bc)}$$

it follows that $(d-a)^2 + 4\,bc > 0$, which is the condition for roots to be real and distinct, so

$\tilde{\mathcal{M}}(z) = \frac{a\,\bar{z}+b}{c\,\bar{z}+d}$ has two fixed points on the real line (horizon).

We previously wrote that by (27) on p. 152, the fixed points of $\tilde{\mathcal{M}}(z)$ must be the same as those of M(z). We based this on the fact that M(z) and $\tilde{\mathcal{M}}$(z) have the same coeffients up to sign. Only the imaginary parts of z are different.

$$\xi_{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2\,c} \qquad (27)$$

If we now substitute our equivalent values of the coefficients into (2), we get

$$\xi_{x\pm} = \frac{((-a)-d) \pm \sqrt{(d-(-a))^2 + 4\,(-b)\,c}}{2\,c} = \frac{-(a+d) \pm \sqrt{(a+d)^2 - 4\,bc}}{2\,c}$$

There is a difference in the first summand in the numerator, so we cannot say the fixed points of $\tilde{\mathcal{M}}$(z) equal those of M(z), even if the fixed points of M(z) are on the horizon.

If L is the h-line whose ends are these two points, explain why L is invariant under $\tilde{\mathcal{M}}$.

L is constructed as an h-line passing through the fixed points of $\tilde{\mathcal{M}}$. As such, it is the unique h-line passing through the fixed points and orthogonal to the horizon at those points. Since $\tilde{\mathcal{M}}$ is an opposite motion, it is a direct motion (M) in L followed by or preceded by a reflection in L: $\tilde{\mathcal{M}} = \mathcal{I}_L \circ$ M. The direct motion M carries points of L to points of L. Reflection in L carries a point to itself. Therefore, L is invariant under $\tilde{\mathcal{M}}$. To illustrate, we can write $\tilde{\mathcal{M}} = \mathcal{I}_L \circ \mathfrak{R}_B \circ \mathfrak{R}_A$, where A and B are h-lines orthogonal to L (Figure 2). The direct motion is accomplished by means of the composition of two reflections $\mathfrak{R}_B \circ \mathfrak{R}_A$, which is equivalent to an h-translation along L. $\mathfrak{R}_B \circ \mathfrak{R}_A$ takes points in L to other points in L. $\mathcal{I}_L$ takes points in L to themselves. Hence L is invariant under $\tilde{\mathcal{M}}$.

(iv) Deduce that $\tilde{\mathcal{M}}$ is always a glide reflection: h-translation along L, followed by (or preceded by) h-reflection in L.

We are concerned with motions of the UHP that map a line segment to another line segment of equal length. By chapter 1, p. 36, (25) we find that there are only two such motions: one direct motion $\mathcal{M}$ and one opposite motion $\tilde{\mathcal{M}}$. Furthermore, (25) states that $\tilde{\mathcal{M}}$ is equivalent to $\mathcal{M}$ followed by reflection in the map segment A'*B'.* In this problem, we have constructed the h-line L on the two fixed points of $\tilde{\mathcal{M}}$. This means that motions of points under $\tilde{\mathcal{M}}$ follow h-lines parallel to L. Then the movement of one segment AB to another segment A'B' always involves a glide (h-translation on L), which can be written

$$\mathcal{M}(z) = \mathcal{R}_D \circ \mathcal{R}_C(z) = \mathcal{T}_{H\{A,\,A'\}}(z)$$

where C and D are h-lines orthogonal to L. [Substitute C and D for A and B in Figure 2.] According to (25), $\tilde{\mathcal{M}}$ also requires reflection in the mapped segment A'*B'.* Hence, $\tilde{\mathcal{M}}$ is always a glide reflection.

[Question deleted.]