Needham, Chapter 6, Exercise 23, p. 336.
G. Palmer, February 10, 2017, 9:15 am, PST

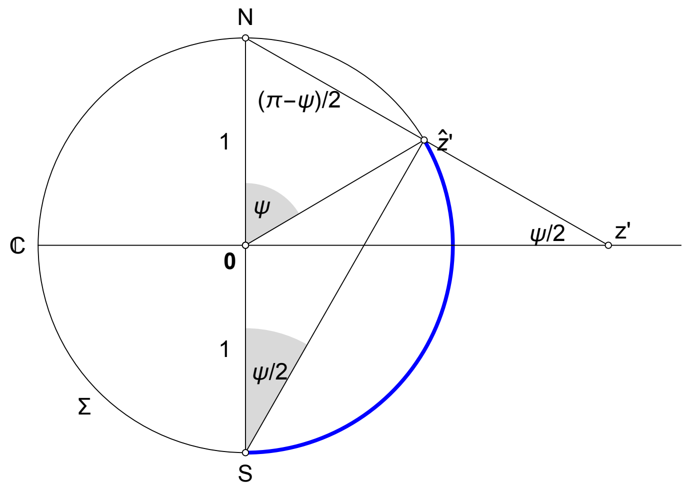

Figure 1. S{a,z} = 2 arctan|(z−a)/($\bar{a}$z+1)|

23 (i) Let $\hat{a}$ and $\hat{z}$ be two point[s] on Σ, and let a and z be their stereographic images in ℂ. If S{a, z} is the distance (on the sphere) between $\hat{a}$ and $\hat{z}$, show that

$$S\{a, z\} = 2\tan^{-1}\left|\frac{z-a}{\bar{a}z+1}\right|.$$

[Hint: Use (20) to bring a to the origin.]

We make use of the scheme presented in figure [23b], p. 147.

By (20), Needham must mean the equation on p. 288, which “represents the most general motion of Σ.”

$$R_a^{\psi}(z) = \frac{Az+B}{-\bar{B}z+\bar{A}} \qquad (20)$$

This is the Möbius representation of the most direct motion of Σ. Write a $R_a^{\psi}(z)$ that takes a to 0 and z to a point z’ in ℂ.

A = 1; B = -a

$$R_a^{\psi}(z) = \frac{z-a}{\bar{a}z+1} = z'$$

This is the expression in the arctan of the problem statement. We use arctan rather than $\tan^{-1}$ to avoid confusion with 1/tan, a mistake to which we are susceptible.

$R_a^{\psi}(z)$ preserves angles and circles because it is a Möbius transformation, but does it preserve distances? Yes, motions are distance preserving mappings (p. 34, Ch 1, § IV, §§ 2 Classifying Motions). We presume this also holds true for Σ and the h-plane. $R_a^{\psi}(z)$ represents a direct motion of Σ, so it must preserve distances on Σ.

Plot a unit circle on a vertical section through N0z'. For convenience, use a θ rotation about the origin to put z' on the real line. Then $R_a^\psi(z) - R_a^\psi(a) = |\frac{z-a}{\bar{a}z+1}| = |z'|$, because $R_a^\psi(a) = a' = 0$.

In order to move a to the origin, we have moved $\hat{a}$ to S, where we can refer to it as $\hat{a}$'. We want to know the length of the blue arc subtended by the angle S0$\hat{z}$' (which is the arc $\hat{a}$'$\hat{z}$'), because it equals the length of the arc $\hat{a}\hat{z}$. We can see that $|z'| = \tan(\pi/2-\psi/2)$ and that the angle S0$\hat{z}$' $= \pi/2 + (\pi/2 - \psi) = \pi - \psi = 2\,(\pi/2-\psi/2) = 2 \arctan |z'|$.

$$S\{a, z\} = r\,(\pi - \psi) = 2 \arctan |z'| = 2 \tan^{-1} \left| \frac{z-a}{\bar{a}z+1} \right|,$$

where r = 1. This is what we wanted.

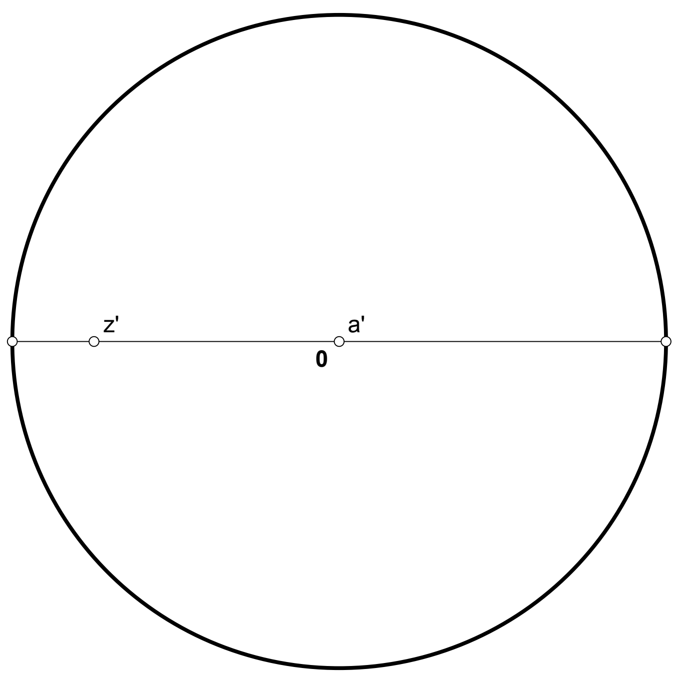

Figure 2. Metric of Poincaré Disc

(ii) Show that the h-distance formula (46) in the Poincaré disc can be reexpressed as

$$H\{a,z\} = 2 \tanh^{-1} \left| \frac{z-a}{\bar{a}z-1} \right| .$$

The formula to be reexpressed is

$$H\{a,z\} = \ln\left(\frac{|\bar{a}z-1|+|z-a|}{|\bar{a}z-1|-|z-a|}\right) \qquad (46)$$

Transform the PD so that a → 0 and z → z' on the real line as in Figure 2, and begin with the prior equation

$$H\{a, z\} = H\{0, M_a(z)\} = H\{0, z'\} = \ln\left(\frac{1+|M_a(z)|}{1-|M_a(z)|}\right)$$

We can write

$$e^{H\{0,M_a(z)\}} = \frac{1+|M_a(z)|}{1-|M_a(z)|}$$

Solve for $|M_a(z)|$. At this point, one might ask, “Why are we solving for $|M_a(z)|$? We have an established equation for $M_a(z)$, namely $M_a(z) = \frac{z-a}{\bar{a}z-1}$, from p. 320.” The reason is that solving this equation for $|M_a(z)|$ gives an equation in terms of a quotient of exponential functions, as one could guess from the LHS. This quotient of exponentials has a tanh equivalent and we seek an equation with arctanh on the RHS.

$$|M_a(z)| = \frac{e^{H\{0,M_a(z)\}}-1}{e^{H\{0,M_a(z)\}}+1} = \tanh(H\{0, M_a(z)\}/2),$$

from the hyperbolic functions in wikipedia:

$\tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}$ <https://en.wikipedia.org/wiki/Hyperbolic_function>.

Then

$$H\{0, M_a(z)\} = 2 \operatorname{arctanh} |M_a(z)|$$

Now substitute the form from p. 320,

$$M_a(z) = \frac{z-a}{\bar{a}z-1}$$

$$H\{a,z\} = H\{0, M_a(z)\} = 2 \operatorname{arctanh} \left|\frac{z-a}{\bar{a}z-1}\right|$$