Needham, Chapter 6, Exercise 22, p. 336.
G. Palmer, February 3, 2017, 10:10 am, PST

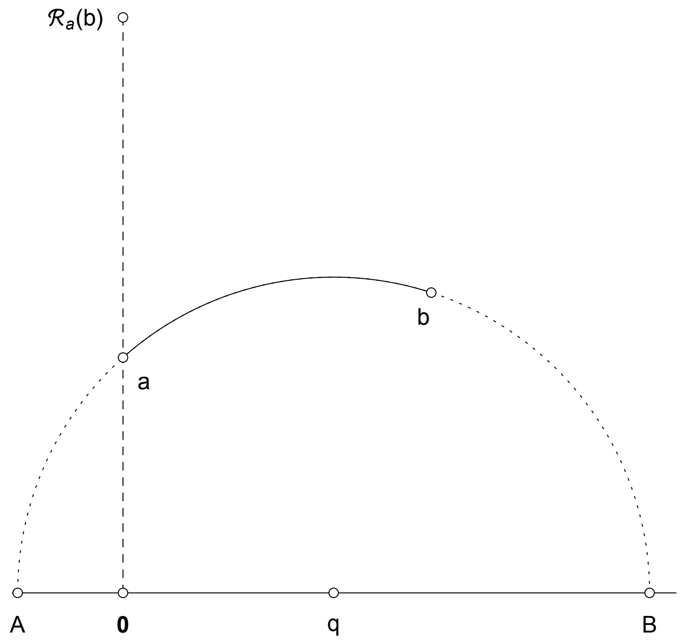

Figure 1. From figure [a], p. 336

22 (i) Referring to figure [a] above, show that the h-separation of two points in the upper half-plane may be expressed in terms of a cross-ratio as

$$H\{a, b\} = \ln[a, B, b, A]$$

[Hint: Apply an h-rotation centred at a to bring b into a position vertically above a.]

Answer

The hint suggests that we need to align a and b on a vertical line, say L. It also suggests we do this by an h-rotation of b centred at a. As I read the hint, it seemed to leave A and B unrotated. Point a is treated as a fixed point, so A and B must be rotated by the same h-rotation that brings b to alignment. The h-rotation would presumably be accomplished by reflections about two lines intersecting at the fixed point a, as in Figure [34a], p. 172]. The double reflection seems unduly complicated here and seems to have little to do with expressing the h-separation in terms of a cross-ratio. Vasco avoided it by simply placing placing a on a line passing through the origin, b on L and taking up work on the cross-ratio. He evidently understood the global effect of the M rotation. In my opinion, it would be better to rewrite the hint as follows: “[Hint: Place a on the Y axis and apply a positive h-rotation centred at a to bring A, a, b, and B into alignment on the axis so that b appears above a.]” This hint tells one immediately that a is a fixed point. It also tells us that we now have three points, a, A, and B that are carried to the three specific points a, 0, and ∞, respectively. With three points carried to another specific three points, one can write a cross-ratio that accomplishes the movement. We now proceed as on P. 154, ℙ 4.

$$R(z) = k\,\frac{z-A}{z-B}$$

$$a = k\frac{a-A}{a-B}, \quad \text{because a is a fixed point, so } k = a\ \frac{a-B}{a-A}$$

Substitute for k in (1)

$$\mathcal{R}_a(z) = a\ \frac{a-B}{a-A}\ \frac{z-A}{z-B}$$

so

$$\mathcal{R}_a(b) = a\ \frac{a-B}{a-A}\frac{b-A}{b-B}$$

$$\frac{\mathcal{R}_a(b)}{a} = \frac{a-B}{a-A}\frac{b-A}{b-B} = [a, B, b, A]$$

The metric of the UHP is $|\ln(\frac{y_1}{y_2})|$, where y_1 and y_2 are the imaginary parts of two points on the y axis (See (35), p. 302). This has the virtue of assuring that $\frac{y_1}{y_2}$ is real. We have shown that the cross ratio has this structure because $\frac{y_b}{y_a} = \frac{\mathcal{R}_a(b)}{a}$ (the i-factors cancel and a = $R_a(a)$), so that we can write

$$H\{a, b\} = H\{b,a\} = |\ln(\frac{\mathcal{R}_a(b)}{a})| = \ln[a, B, b, A].$$

This is what we wanted. The abolute brackets are not needed in this instance, because a and b are positive reals and b > a, so the log is positive.

Note:

We are apparently mapping b to i, rather than to the 1 used on p. 154. Is this legitimate? Have we rotated the cross ratio by Pi/2 by putting ∞ on the y axis?

For further confirmation of the positive sign of the cross ratio, see the following note, which is not part of the answer.

We can express [a,B,b,A] in terms of $re^{i\theta}$:

Angle equations:

arg(a-A) = arg(a-B)-Pi/2

arg(b-B) = Pi/2+arg(b-A)

$$[a,B,b,A] = \frac{|a-B||b-A|\, e^{i(\arg(a-B)+\arg(b-A))}}{|a-A||b-B|\, e^{i(\arg(a-B)-\mathrm{Pi}/2+\mathrm{Pi}/2+\arg(b-A)}} = \frac{|a-B||b-A|}{|a-A||b-B|}$$

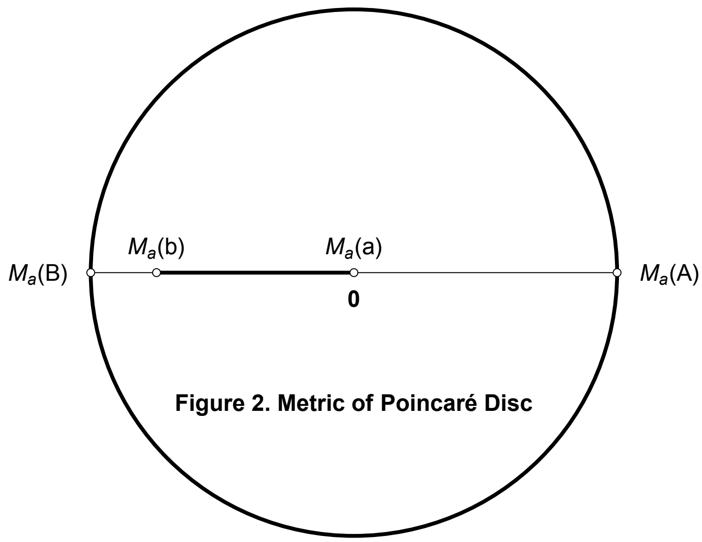

Figure 2. Metric of Poincaré Disc

(ii) Show that the same formula applies to the Poincaré disc in figure [b].

We want to align A, a, b, B in an h-line that is a conveniently horizontal diameter in the PD (Figure 2), and we want to show that the metric of the PD can be rewritten as the cross-ratio. The metric formula we will use is

$$H\{0, M_a(z)\} = \ln\left(\frac{1+M_a(z)}{1-M_a(z)}\right), \quad \text{p. 321, ℙ 2} \qquad (1)$$

To align A, a, b, B, we can write a cross-ratio that sends B to -1, A to 1, and a to 0. Then $M_a(b)$ must fall on the real line somewhere between $M_a(a)$ and $M_a(B)$. We plot the points on the real line, use the same letters. To make the algebra more convenient, we now let a = $M_a(a)$, $b = M_a(b)$, $A = M_a(A)$, $B = M_a(B)$.

Given from the note following (i):

$$[a,B,b,A] = \frac{|a-B||b-A|}{|a-A||b-B|}$$

Then for this rotation (Figure 2),

$$[a,B,b,A] = \frac{|-B||b-A|}{|-A||b-B|} = \frac{|b-A|}{|b-B|} \qquad (2)$$

$$H\{a,b\} = H\{0, M_a(b)\} = \ln\left(\frac{1+|M_a(b)|}{1-|M_a(b)|}\right) = \ln\left(\frac{1+|1-|b-B||}{1-|1-|b-B||}\right) \qquad (3)$$

$$= \ln\left(\frac{2-|b-B|}{|b-B|}\right) = \ln\left(\frac{|b-A|}{|b-B|}\right) = \ln[a, B, b, A] \qquad \text{sub. LHS from (2)}$$

This is what we wanted. We can write (3) because the h-distance between a and b remains constant under the Möbius transformation and because the distance formula can be written as in (1).

But Vasco’s much quicker answer seems sufficient

Since we know from page 316 paragraph one that the h-lines of the Poincaré disc are precisely the images of h-lines in the Poincaré upper half-plane, and that the h-separation of two points in the disc is defined to be the h-separation of their preimages in the upper half-plane then the formula in part (i) applies also to the Poincaré disc in figure (b).

And in fact one can see by inspection of the Poincaré disc that if [a, B, b, A] applies to an h-line in the UHP, it should also apply to an h-line in the PD, since both are in the hyperbolic plane.