Needham, Chapter 6, Exercise 21, p. 335.
G. Palmer, January 22, 2017, 7:45 am, PST

21 In the upper half-plane, the h-rotation $\mathcal{R}_i^{\phi}$ through angle ϕ about the point i is given by the following Möbius transformation:

$$\mathcal{R}_i^{\phi}(z) = \frac{cz+s}{-sz+c}, \text{ where } c = \cos(\phi/2) \text{ and } s = \sin(\phi/2).$$

Prove this in three ways:

(i) Show that $\mathcal{R}_i^{\phi}(i) = i$ and $\{\mathcal{R}_i^{\phi}\}'(i) = e^{i\phi}$. Why does this prove the result?

Given: $(-sz + c)^2 = c^2 - s^2 - 2csi$ where z = i (1)
$\sin\phi = 2\sin\phi/2\cos\phi/2$ (2)
$\cos\phi = \cos^2\phi/2 - \sin^2\phi/2$ (3)

$$\mathcal{R}_i^{\phi}(\mathrm{i}) = \frac{ci+s}{-si+c} = \frac{ci+\sqrt{1-c^2}}{-\sqrt{1-c^2}\,i+c}$$

$$= \frac{\sqrt{1-c^2}+ci}{c-\sqrt{1-c^2}\,i} = i\,\frac{\sqrt{1-c^2}+ci}{i\left(c-\left(\sqrt{1-c^2}\,i\right)\right)} = i\,\frac{\sqrt{1-c^2}+ci}{ci+\sqrt{1-c^2}} = i$$

$$\{\mathcal{R}_i^{\phi}\}'(\mathrm{i}) = \left(\frac{cz+s}{-sz+c}\right)'(\mathrm{i}) = (cz+s)'(-sz+c)^{-1} + (cz+s)\left((-sz+c)^{-1}\right)'$$

$$= c/(-sz+c) + s(cz+s)\big/(-sz+c)^2$$

$$= \left(c(-sz+c) + \left(csz + s^2\right)\right)\big/(-sz+c)^2$$

$$= [-csz + c^2 + csz + s^2)]/(-sz+c)^2$$

$$= \left(c^2 + s^2\right)\big/(-sz+c)^2$$

$= 1/(\left(c^2 - s^2\right) - 2csi)$ substituting (1)

$= 1/(\cos\phi - i\sin\phi)$ by (2) and (3)

$= 1/(\cos(-\phi) + i\sin(-\phi))$

$= 1\big/e^{-i\phi} = e^{i\phi}$ by Euler's equation

This demonstrates part (i). (ia) proves the result because a rotation of a point about itself is just the point. (ib) proves the result because the amplitwist of a rotation of a point about itself is just the twist, which is equal to the rotation angle ϕ.

Vasco argues that this explanation is insufficient, which is a point that I can't refute. On p. 164, Ch 3, § VII, §§ 2, we find

We call M an elliptic Möbius transformation if $\tilde{M}$ is elliptic, meaning that the latter is a pure rotation corresponding to m = $e^{i\alpha}$. Since $\tilde{M}$ is a rotation if and only if it maps each origin-centred circle to itself, M(z) is elliptic if and only if it maps each C_2 circle to itself.

Using this information is not straightforward, because $\mathcal{R}_i^{\phi}$(i) derives from M, not from $\tilde{M}$. The fact that $\mathcal{R}_i^{\phi}$(i) = i only tells us that there is a fixed point in the UHP. We must put this information together with $\{\mathcal{R}_i^{\phi}\}'(i) = e^{i\phi}$. This tells us that an infinitesimal vector at i in the UHP is twisted (rotated) by ϕ radians but not expanded or contracted by $\mathcal{R}_i^{\phi}$.

On p. 167, §§ 3, we have

The local effect of M (in an infinitesimal neighborhood of ξ_+) is a rotation centred at ξ_+ through angle ($\pi/3$)—this is the meaning of the multiplier $m_+ = e^{i(\pi/3)}$ associated with ξ_+.

This sentence reveals that the multiplier of an elliptic M is equivalent to the twist of the derivative of M at ξ_+. The complex derivative of an analytic function is also defined as the amplitwist of a vector centred at a point in an infinitesimal neighborhood (Pp. 194-198, Ch 4, § IV). Since M is an analytic function, the multiplier must be the complex derivative, or at least the twist. In the case of $\mathcal{R}_i^{\phi}$, it is clearly the twist, as the amplification equals 1.

How does this prove the result $\mathcal{R}_i^{\phi}(z) = \frac{cz+s}{-sz+c}$, where c = cos($\phi$/2) and s = sin($\phi$/2)? It says that $\mathcal{R}_i^{\phi}$(z) should not expand or contract z, so $\mathcal{R}_i^{\phi}$(z) must map each C_2 circle to itself, it must map $\mathcal{R}_i^{\phi}$(i) to itself, as it does, it must be a pure rotation, and the multiplier must be m = $\{\mathcal{R}_i^{\phi}\}'(i) = e^{i\phi}$, as it is.

The discussion on pp. 319-321 appears relevant to what follows. On p. 321, Needham states that "the Euclidean rotation z ↦ $e^{i\phi} z$ represents the h-rotation $\mathcal{R}_0^{\phi}$" [in the PD]. We know that 0 in the PD corresponds to i in the UHP (p. 316, ℙ 2), so the corresponding h-rotation in the UHP must be $\mathcal{R}_i^{\phi}$. $\mathcal{R}_i^{\phi}$(i) = i proves the result. When we eventually learn that the PD is a Möbius transformation of the UHP, we can say that $\mathcal{R}_i^{\phi}$(i) = i must be true to preserve angles.

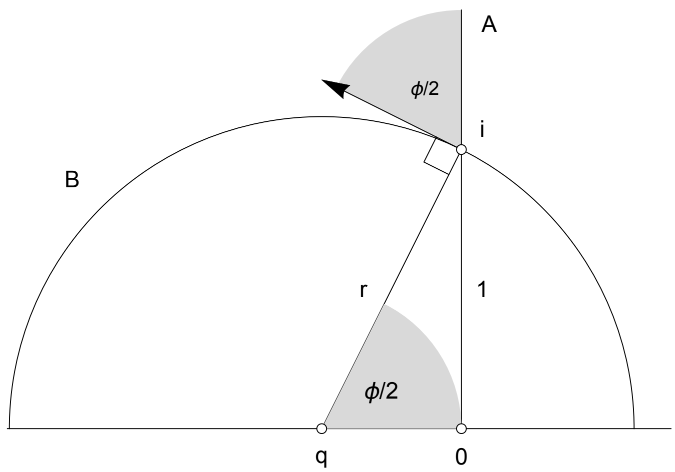

Figure 1. Reflections in Poincaré UHP

(ii) Use the formula for inversion [(4), p. 125] to calculate the composition $\mathcal{R}_i^{\phi} = (\mathcal{R}_B \circ \mathcal{R}_A)$, where A and B are h-lines through i, and the angle from A to B is $\phi/2$. [Hint: Take A to be the imaginary axis, and use a diagram to show that the semicircle B has centre -(c/s) and radius (1/s).]

A is on the imaginary axis, so the lower end of A is situated at the origin (Figure 1). B is another h-line passing through i, which requires that it also be a semicircle orthogonal to the horizon. By construction, q is the centre of B.

Given the notation in Figure 1, we can write

$$s = 1/r \quad \Longrightarrow \quad r = 1/s$$

The radius of B is 1/s.

$$c = |q|/r \quad \Longrightarrow \quad |q| = c\, r = c/s$$

so q must be -c/s, because c, r, and s are all positive reals.

The center of B is -c/s.

Calculate the reflection in B:

$$\mathcal{I}_B(z) = \frac{r^2}{\bar{z}-(-c/s)} + \left(-\frac{c}{s}\right)$$

$$= \frac{(1/s)^2}{\bar{z}+c/s} - \frac{c}{s}$$

$$= \frac{1}{s^2(\bar{z}+c/s)} - \frac{c}{s}$$

$$= \frac{1}{s(s\,\bar{z}+c)} - \frac{c(s\,\bar{z}+c)}{s(s\,\bar{z}+c)}$$

$$= \frac{1-\mathrm{cs}\,\bar{z}-c^2}{s(s\,\bar{z}+c)} = \frac{-\mathrm{cs}\,\bar{z}+1-c^2}{s(s\,\bar{z}+c)} = \frac{-\mathrm{cs}\,\bar{z}+s^2}{s(s\,\bar{z}+c)}$$

$$= \frac{-c\,\bar{z}+s}{s\,\bar{z}+c}$$

A reflection in A is a simple reflection across the vertical line. $\mathcal{R}_A(x + iy) = -x+iy$. Now apply $\mathcal{I}_B(z)$.

$$\mathcal{R}_i^{\phi} = \mathcal{R}_B \circ \mathcal{R}_A = \mathcal{I}_B \circ \mathcal{R}_A(x + iy) = \mathcal{I}_B(-x + iy)$$

$$= \frac{-c(\overline{-x+iy})+s}{s(\overline{-x+iy})+c} = \frac{-c(-x-iy)+s}{s(-x-iy)+c}$$

$$= \frac{cz+s}{-sz+c}$$

This is what we wanted.

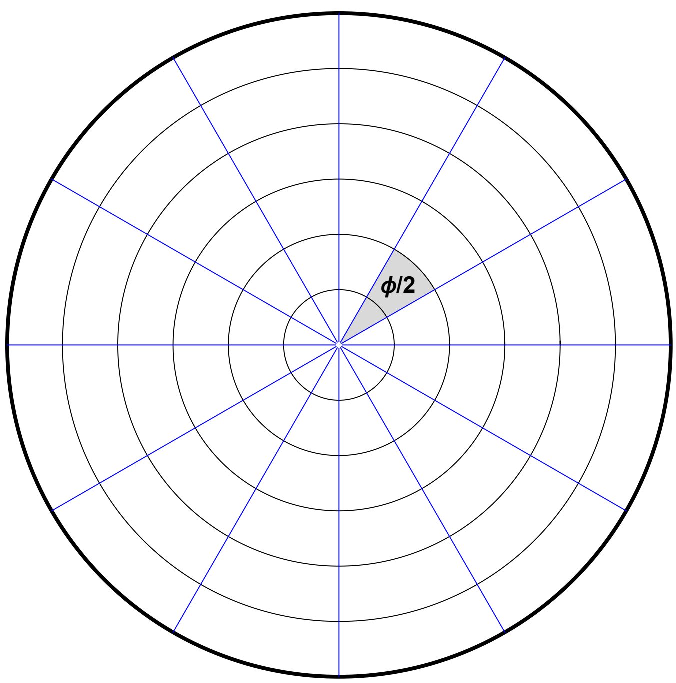

Figure 2. Rays and h−circles on Poincaré disc

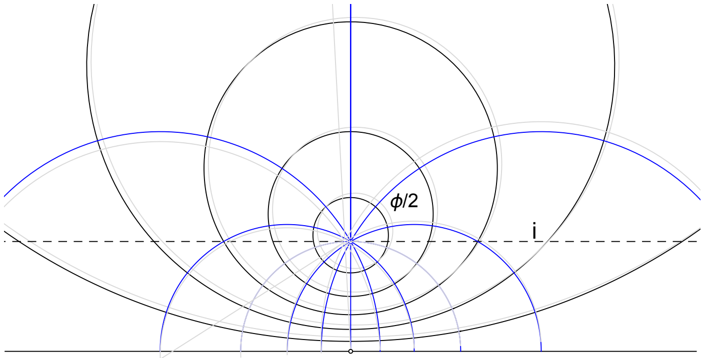

Figure 3. Upper Half–Plane (D^{-1}(PD))

(iii) Describe and explain the geometrical effect of applying ($D \circ \mathcal{R}_i^{\phi} \circ D^{-1}$) to the Poincaré disc, where D is the mapping (42) from the upper-half plane to the Poincaré disc. Deduce that

$$(D \circ \mathcal{R}_i^{\phi} \circ D^{-1})(z) = e^{i\phi} z.$$

Re-express this equation in terms of products of Möbius matrices, and solve for the matrix [$\mathcal{R}_i^{\phi}$].

Answer:

($D \circ \mathcal{R}_i^{\phi} \circ D^{-1}$) first, under D^{-1}, takes the PD to the UHP. Under $\mathcal{R}_i^{\phi}$, the hyperbolic plane undergoes a rotation of ϕ about i, and finally D takes the result back to the PD. At this point I tried calculating the compound function both manually and with *Mathematica*, but I could not see a way to get to $e^{i\phi} z$. Using the equations for D and $\mathcal{R}_i^{\phi}$, and using

$$D^{-1} = \frac{iz-1}{-z+i}$$

the result was

((1+i) (c+i s) z)/(i s (-1+z)+c (1+z)) (4)

I tried working back from $e^{i\phi} = \cos\phi + i \sin\phi$ using (2) and (3), but their patterns don't appear in (4). So I consulted Vasco's answer, which uses an approach based on geometry. I try to reconstruct it here:

In the PD, arrange concentric circles and rays around the origin. The rays are diameters of the PD. The angle between two rays is $\phi/2$ (Figure 2).

When D^{-1} is applied to the PD, the rays go to h-lines (semicircles and a vertical line if a diagonal happens to pass through i and -i on the PD) orthogonal to the horizon of the UHP (Figure 3). Since the angles between rays at the origin of the PD are all $\phi/2$ by construction and D^{-1} is conformal, they must intersect at angles of $\phi/2$. Since $D^{-1}(0) = i$, the h-lines derived from rays in the PD must intersect at the point i in the UHP.

Under D^{-1}, the concentric circles in the PD must go to h-circles centred at i in the UHP; i is not the Euclidean centre. It appears that each circle has its own Euclidean centre given by the equation center = i y cosh(ρ), where y is the y value of its h-centre and ρ is its h-radius (p. 309).

The UHP undergoes a rotation of ϕ about i, which maps the image to itself. The circles map to themselves and ray number n maps to ray number n + 2 at the angular distance of ϕ. In figure 3, the gray lines represent the (slightly offset) images of the circles and h-lines under $\mathcal{R}_i^{\phi}$. Their configuration is consistent with this interpretation, except for the one straight line emanating from i at an angle of approximately 5Pi/4. It is probably an artifact of the plotting function. On mapping the UHP back to the PD using D(z), the PD appears unchanged, but everything is rotated by ϕ, so the result is $e^{i\phi}z$.

Re-express this equation [$(D \circ \mathcal{R}_i^{\phi} \circ D^{-1})(z) = e^{i\phi} z$] in terms of products of Möbius matrices, and solve for the matrix [$\mathcal{R}_i^{\phi}$].

$$D(z) = \frac{iz+1}{z+i} \equiv \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

$$D^{-1}(z) = \frac{iz-1}{-z+i} \equiv \begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix}$$

$$\mathcal{R}_i^{\phi}(z) = \frac{cz+s}{-sz+c} \equiv \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$[e^{i\phi}z] = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}$$

We substitute these into the equation:

$$[D] \,.\, [\mathcal{R}_i^{\phi}] \,.\, [D^{-1}] = [e^{i\phi}z]$$

$$[\mathcal{R}_i^{\phi}] \,.\, [D^{-1}] = [D^{-1}] .\, [e^{i\phi}z]$$

$$[\mathcal{R}_i^{\phi}] = [D^{-1}].[e^{i\phi}z].[D]$$

$$= \begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

$$[\mathcal{R}_i^{\phi}] = \begin{pmatrix} -1 - e^{i\phi} & i - ie^{i\phi} \\ -i + ie^{i\phi} & -1 - e^{i\phi} \end{pmatrix} \qquad (5)$$

$$\equiv \frac{\left(-1-e^{i\phi}\right) z + i - ie^{i\phi}}{\left(-1+e^{i\phi}\right) iz - 1 - e^{i\phi}}$$

This seems to be a reasonable solution for [$\mathcal{R}_i^{\phi}$], but most likely the preferred solution looks like (19) on

p. 288:

$$[\mathcal{R}_a^{\psi}] = \begin{pmatrix} e^{i(\Psi/2)} \mid a \mid^2 + e^{-i(\Psi/2)} & 2\,\text{ia}\sin(\Psi/2) \\ 2\,i\,\overline{a}\sin(\Psi/2) & e^{-i(\Psi/2)} \mid a \mid^2 + e^{i(\Psi/2)} \end{pmatrix},$$

where a and $-(1/\overline{a})$ are fixed points and the multiplier m = $e^{-\mathrm{i}\Psi/2}$ associated with a. In our problem, the multiplier is $e^{\mathrm{i}\phi}$, and i is a fixed point of $[\mathcal{R}_a^{\psi}]$ by part (i). Then, by substitution,

$$[\mathcal{R}_i^{\phi}] = \begin{pmatrix} e^{i(\phi/2)} \mid i \mid^2 + e^{-i(\phi/2)} & 2\,i^2\sin(\phi/2) \\ 2\,i(-i)\sin(\phi/2) & e^{-i(\phi/2)} \mid i \mid^2 + e^{i(\phi/2)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i(\phi/2)} + e^{-i(\phi/2)} & -2\sin(\phi/2) \\ 2\sin(\phi/2) & e^{-i(\phi/2)} + e^{i(\phi/2)} \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos(\phi/2) & -2\sin(\phi/2) \\ 2\sin(\phi/2) & 2\cos(\phi/2) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \text{ by omitting constant 2}$$

But this gives $\mathcal{R}_i^{\phi} = \frac{\mathrm{cz}-s}{\mathrm{sz}+c}$, which is not the equation we were given in exercise (i). The signs of s and sz are inverted. I'm not sure why. The calculation is not complicated. It may be that $[\mathcal{R}_a^{\psi}]$ on p. 288 is actually for a negative (clockwise) rotation, as is suggested in Figure [15]. That would preserve the sign of c while changing the sign of s.

Vasco has shown that the form of $[\mathcal{R}_i^{\phi}]$ can also be derived directly from (5), somewhat as follows:

$$[\mathcal{R}_i^{\phi}] = \begin{pmatrix} -1 - e^{\mathrm{i}\phi} & i - \mathrm{ie}^{\mathrm{i}\phi} \\ -i + \mathrm{ie}^{\mathrm{i}\phi} & -1 - e^{\mathrm{i}\phi} \end{pmatrix}$$

$$= \begin{pmatrix} -\left(1 + e^{\mathrm{i}\phi}\right) & -\left(\mathrm{ie}^{\mathrm{i}\phi} - i\right) \\ -\left(i - \mathrm{ie}^{\mathrm{i}\phi}\right) & -\left(1 + e^{\mathrm{i}\phi}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -e^{\mathrm{i}\phi/2}\left(e^{-\mathrm{i}\phi/2} + e^{\mathrm{i}\phi/2}\right) & -e^{\mathrm{i}\phi/2}\left(\mathrm{ie}^{\mathrm{i}\phi/2} - \mathrm{ie}^{-\mathrm{i}\phi/2}\right) \\ -e^{\mathrm{i}\phi/2}\left(\mathrm{ie}^{-\mathrm{i}\phi/2} - \mathrm{ie}^{\mathrm{i}\phi/2}\right) & -e^{\mathrm{i}\phi/2}\left(e^{-\mathrm{i}\phi/2} + e^{\mathrm{i}\phi/2}\right. \end{pmatrix}$$

$$= -e^{\mathrm{i}\phi/2} \begin{pmatrix} 2\cos(\phi/2) & 2\,i\sin(\phi/2)/i \\ -2\,i\sin(\phi/2)/i & 2\cos(\phi/2) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \quad \text{by omitting } -2\,e^{\mathrm{i}\phi/2}$$

This is equivalent to $\mathcal{R}_i^{\phi} = \frac{\mathrm{cz}+s}{-\mathrm{sz}+c}$. This is the correct form.