Needham, Chapter 6, Exercise 20, p. 335.
G. Palmer, January 17, 2017, 4:30 pm, PST

20 Consider the mapping z ↦ w = $M_a^{\phi}(z)$ of the Poincaré disc to itself. Use the calculational approach of the previous exercise to show that z ↦ w is a hyperbolic motion, i.e. it preserves the metric

$$\frac{2|\mathrm{dw}|}{1-|w|^2} = \mathrm{d}\hat{s} = \frac{2|\mathrm{dz}|}{1-|z|^2}.$$

Given:

$$M_a^{\phi}(z) = e^{\mathrm{i}\phi}\left(\frac{z-a}{\bar{a}z-1}\right)$$ (p. 178), where a is an interior point on the PD (p. 111, Ex. 3)

$$|\,\bar{a}z-1\,|^2 = (\bar{a}z-1)(a\bar{z}-1) = |\,a\,|^2\,|\,z\,|^2 - \bar{a}z - a\bar{z} + 1$$

Thus, w = $M_a^{\phi}(z) = e^{\mathrm{i}\phi}\left(\frac{z-a}{\bar{a}z-1}\right)$. We treat $e^{\mathrm{i}\phi}$ as a constant with $\phi = 0$ and use $M_a(z) = \frac{z-a}{\bar{a}z-1}$. Then

$$|\mathrm{dw}| = \left|\frac{d\,M_a(z)}{\mathrm{dz}}\mathrm{dz}\right| = |\,M_a'(z)\,\mathrm{dz}|$$

$$= \left|\frac{|a|^2-1}{(\bar{a}z-1)^2}\mathrm{dz}\right| = \frac{|-(1-|a|^2)|}{|\bar{a}z-1|^2}|\mathrm{dz}|$$

$$= \frac{1-|a|^2}{|\bar{a}z-1|^2}|\mathrm{dz}| \qquad \text{because } |\,a\,|^2 < 1$$

$$|\,w^2\,| = |M_a(z)\,|^2 = \left|\frac{z-a}{\bar{a}z-1}\right|^2$$

$$= \frac{z-a}{\bar{a}z-1}\,\overline{\frac{z-a}{\bar{a}z-1}} = \frac{|z|^2-\bar{a}z-a\bar{z}+|a|^2}{|\bar{a}z-1|^2}$$

$$1-|\,w^2| = 1-\frac{|z|^2-\bar{a}z-a\bar{z}+|a|^2}{|\bar{a}z-1|^2} = \frac{|\bar{a}z-1|^2}{|\bar{a}z-1|^2} - \frac{|z|^2-\bar{a}z-a\bar{z}+|a|^2}{|\bar{a}z-1|^2}$$

$$= \frac{(|a|^2|z|^2-\bar{a}z-a\bar{z}+1)-(|z|^2-\bar{a}z-a\bar{z}+|a|^2)}{|\bar{a}z-1|^2}$$

$$= \frac{|a|^2|z|^2+1-|z|^2-|a|^2}{|\bar{a}z-1|^2} = \frac{(|a|^2-1)|z|^2-(|a|^2-1)}{|\bar{a}z-1|^2}$$

$$= \frac{(1-|a|^2)(1-|z|^2)}{|\bar{a}z-1|^2}$$

Now construct the metric by substitution for |dw| and 1 – $|\,w\,|^2$.

$$\frac{2|\mathrm{dw}|}{1-|w|^2} = \frac{2(1-|a|^2)|\mathrm{dz}|}{|\bar{a}z-1|^2}\,\frac{|\bar{a}z-1|^2}{(1-|a|^2)(1-|z|^2)}$$

$= \frac{2|dz|}{1-|z|^2}$, by cancellation

so the metric is preserved.