Needham, Chapter 6, Exercises 2, p. 328
G. Palmer, September 8, 2016, 3:30 PM PST

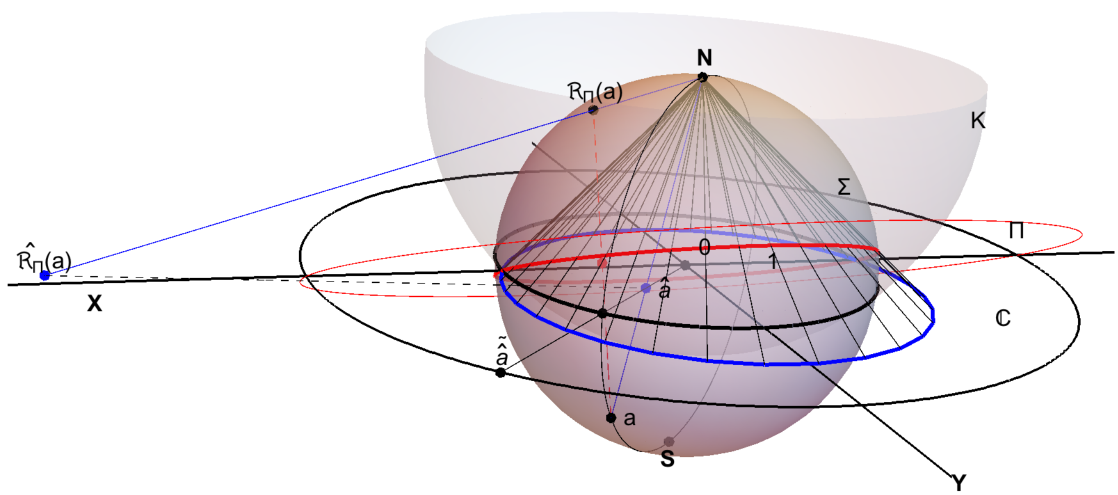

Figure 1. Inversions in K induce stereographic projections that are restricted to Σ and ℂ

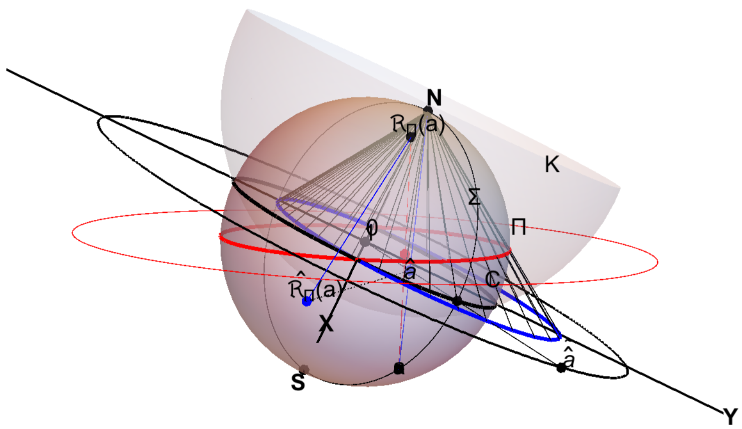

Figure 2. Inversions in K induce stereographic projections that are restricted to Σ and ℂ

2 Explain (18) [p. 287] *by generalizing the argument that was used to obtain the special case (17), on p. 143. That is, think of reflection of the sphere in terms of reflection of space in a plane Π, as in [8], p. 280. Also, think of stereographic projection as the restriction to the sphere* [Σ] *of the three dimensional inversion I_K, where K is the sphere of radius $\sqrt{2}$ centred at the north pole of Σ (see [13b]). Now let a be a point on Σ, and consider the effect of $\mathcal{I}_K$ on a, Π, and $\mathcal{R}_\Pi(a)$.*

We let $\mathcal{R}_\Pi$ stand for a reflection in the plane Π. Figures 1 and 2 are identical except for different viewpoints.

(a) The effect of $\mathcal{I}_K$ on a in Σ is to induce $\hat{a}$ (blue point) in the complex plane ℂ by stereographic projection. We can apply $\mathcal{I}_{\mathbb{C}}(\hat{a})$ to obtain the reflection $\hat{a}$. Conversely, the effect of $\mathcal{I}_K$ on ℂ is to send all points in ℂ to Σ. The point $\hat{a}$ will figure in part (c).

(b) If Π is allowed to be a plane passing through the origin at any angle to the N-S axis, then Π restricted to Σ would create a great circle line (inner red circle in Figure 1). Π was created by rotating ℂ by Pi/8 about the X axis. Blue is used to indicate points and objects that are stereographically induced (but they may also be induced by inversion in a circle on $\mathbb{C}$, as with $\hat{a}$). The cone is composed of the projection lines induced by projecting Π to $\mathbb{C}$. Then the effect of $I_K(\Pi)$ is to induce a conformal (direction preserved) circle in $\mathbb{C}$ (See pp. 140-142, *Stereographic Projection*). Notice that the induced blue circle in $\mathbb{C}$ is not coincident with the unit circle in $\mathbb{C}$ (black).

(c) $\mathcal{I}_K \circ \mathcal{R}_\Pi(a)$ induces reflection (inversion) of $\mathbb{C}$ in the blue circle $\mathcal{I}_K(\Pi)$. The blue point $\hat{\mathcal{R}}_\Pi(a)$ represents $\mathcal{I}_K \circ \mathcal{R}_\Pi(a)$. The point $\hat{a}$, itself induced by a, is the reflection of $\hat{\mathcal{R}}_\Pi(a)$ in the induced blue circle. The claim is that

$$\mathcal{I}_K \circ \mathcal{R}_\Pi = \mathcal{I}_{\mathcal{I}_K(\Pi)} \circ \mathcal{I}_K$$

The two reflections $\mathcal{I}_K$ and $\mathcal{R}_\Pi$ do not equal a conformal Möbius transformation, because $\mathcal{I}_K$ is a conformal inversion, but $\mathcal{R}_\Pi(a)$ is not. The RHS is an inversion in a circle applied to a stereographic inversion, so it also appears to be nonconformal. It appears that the fact that the nonconformality of the inversion in the blue circle is not affected by the fact that the circle is itself a stereographic projection in $\mathbb{C}$.

Afterward: After reading Vasco's answer, I see that I missed the point. *Vasco wrote: Now apply the three-dimensional preservation of symmetry result (13).* [p. 135]. *Since a and $\mathcal{R}_\Pi(a)$ are symmetric with respect to Π, their stereographic images $\mathcal{I}_K(a)$ and $\mathcal{I}_K(\mathcal{R}_\Pi(a))$ will be symmetric with respect to the stereographic image of $\mathcal{I}_K(\Pi)$.* I think what I wrote above was not incorrect, but I had forgotten the wording of the three-dimentional symmetry principle. My reasoning was more along these lines: Since all the operations that produced $\hat{a}$ and $\hat{\mathcal{R}}_\Pi(a)$ were symmetric, then these two points lying on opposite sides of the induced blue circle in $\mathbb{C}$ must be symmetric. When I tested the idea empirically, I found that they are indeed.

Notes:

Table of reflections

reflection in a line	reflection in a plane	reflection in a great circle
inversion in a circle	inversion in a sphere (including stereographic projection)	