Needham, Chapter 6, Exercise 19, p. 335.
G. Palmer, January 17, 2017, 2:40 pm, PST

19 To derive the metric (44), consider the mapping (42) $z \mapsto w = D(z)$ from the upper half-plane to the Poincaré disc. An infinitesimal vector dz emanating from z is amplitwisted to an infinitesimal vector $dw = D'(z)\, dz$ emanating from w, and (by definition) the h-length $d\hat{s}$ of dw is the h-length of dz. Verify (44) by showing that

$$\frac{2|\mathrm{dw}|}{1-|w|^2} = \frac{|\mathrm{dz}|}{\operatorname{Im} z} = \mathrm{d}s$$

Given:

$$\mathrm{D(z)} = \frac{\mathrm{iz}+1}{z+i} \qquad (42)$$

$$\mathrm{d}s = \frac{2}{1-|z|^2}\mathrm{ds} \qquad (44)$$

$$\mathrm{z = x + y\, i}$$

Then

$$\mathrm{w} = \mathrm{D(z)} = \frac{\mathrm{iz}+1}{z+i}$$

$$\mathrm{dw} = \frac{\mathrm{dw}}{\mathrm{dz}}\,\mathrm{dz} = \mathrm{D'(z)\, dz} = -\frac{2\,\mathrm{dz}}{(z+i)^2}$$

$$|w| = \left|\frac{1+i\, z}{i+z}\right|$$

$$\frac{2|\mathrm{dw}|}{1-|w|^2} = \frac{2\left|-\frac{2\,\mathrm{dz}}{(z+i)^2}\right|}{1-\left|\frac{1+i\,z}{i+z}\right|^2} = \frac{4|\mathrm{dz}|}{\left(1-\left|\frac{1+i\,z}{i+z}\right|^2\right)\left|i+z\right|^2}$$

$$= \frac{4|\mathrm{dz}|}{\left|i+z\right|^2-|1+i\,z|^2} = \frac{4|\mathrm{dz}|}{(i+z)\left(-i+\bar{z}\right)-(1+\mathrm{iz})\left(1-i\,\bar{z}\right)}$$

$$= \frac{4|\mathrm{dz}|}{1+i\,\bar{z}-\mathrm{iz}+|z|^2-\left(1-i\,\bar{z}+\mathrm{iz}+|z|^2\right)}$$

$$= \frac{4|\mathrm{dz}|}{2\,i\left(\bar{z}-z\right)} = \frac{2|\mathrm{dz}|}{i(-2\,\mathrm{iy})} = \frac{|\mathrm{dz}|}{y} = \frac{|\mathrm{dz}|}{\operatorname{Im} z}$$

So

$$\frac{2\,|\mathrm{dw}|}{1-|w|^2} = \frac{|\mathrm{dz}|}{\operatorname{Im} z} = \mathrm{d}s$$

We now have the expression ds = $\frac{2\,|\mathrm{dw}|}{1-|w|^2}$, but w corresponds to z in Figure [36] and (44), so ds must equal |dw|. This verifies (44). Vasco remarked “It would have been less confusing if Needham had used w rather than z in figure 36b to avoid confusion with the z in figure 36a.” We agree.