Needham, Chapter 6, Exercise 18, p. 335.
G. Palmer, January 15, 2017, 12 midnight, PST

Out[374]=

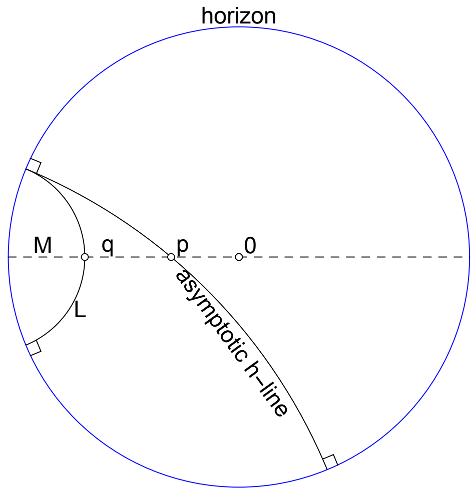

Figure 1. Poincaré Disc (Blue) with asymptotic h–lines

Out[375]=

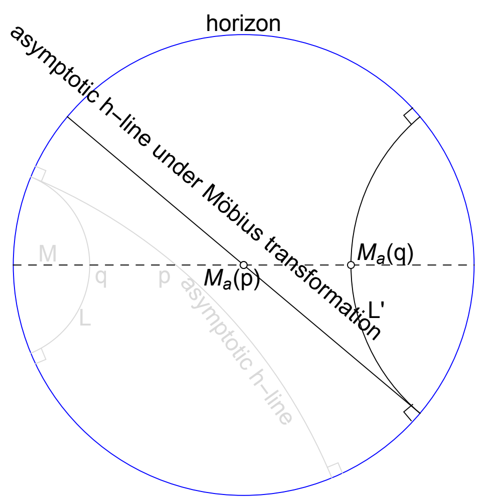

Figure 2. Poincaré Disc (Blue)
under Möbius transformation

Out[376]=

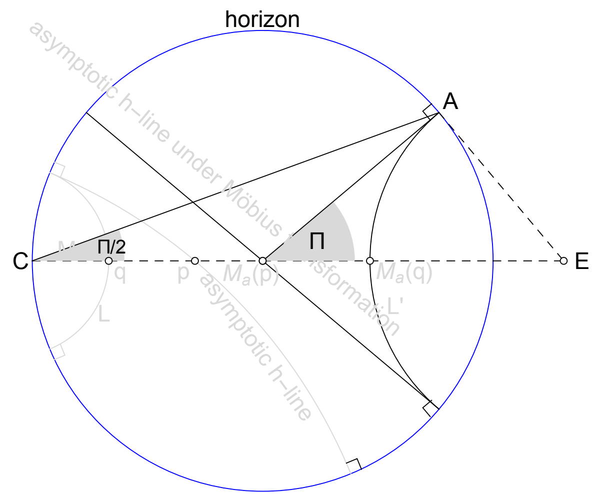

Figure 3. Angle of parallelism, Poincaré Disc (Blue)

18 Use the Poincaré disc to rederive the formula tan(Π/2) = e^{-D} for the angle of parallelism. [Hint: Let one of the h-lines be a diameter.]

This solution is based on a reading of Vasco's solution, though my intent is to reconstruct a valid solution without copying his. Following Figure [26], we use "M" for the h-line passing through p that cuts L at a right angle, and "L" for an h-line with an asymptote (Figure 1). The asymptote is just labeled "asymptotic." We have plotted only one. We use "p" for a point at the intersection of M and the asymptotic, and "q" for the intersection of L and M. This choice of labels follows the function of the h-lines in Figure [26], though it may not appear so, because the PD requires a different layout. Since the horizon of the PD is circular, we can place M and L at any orientation. It is convenient to place them so that M lies on the real axis, because this simplifies calculation of the Möbius transformation and ultimately, it simplifies the trigonometry.

Now we must transform the curved asymptotic h-line into a straight asymptotic line by means of a direct motion of the disc. We do this by moving p to the origin with the Möbius transformation $M_a^{\phi}(z)$ = $e^{i\phi} M_a(z)$, where

$$M_a(z) = \frac{z-a}{\bar{a}z-1},$$

a is any point inside the disc, and ϕ is a rotation about the origin. If we choose p to instantiate a, then we have

$$M_p(z) = \frac{z-p}{\bar{p}z-1} = \frac{z-p}{pz-1}, \text{ because p is real by construction.}$$

We find that $M_p(p) = 0$. Since the asymptotic h-line must pass through p, it must be the unique h-line from its intersection point with L at a point on M that passes through the origin, so it must be a diameter (Figure 2). Since Möbius transformations are conformal the angles of the asymptotic with the PD and with L and M remain the same. It appears that we have simply bent the asymptotic h-line into a more convenient shape. We must also apply $M_p(z)$ to the h-line L, which results in L', which makes the same angles in the same orientations with M and the asymptotic. Since the asymptotic h-line is now straight, it is possible to apply ordinary trig identities. In Figure 3, we feel confident drawing the line from the origin to A to represent the Möbius transformation of a second asymptotic if we had plotted one at the second point of intersection of L and the horizon. We could understand the line to be a reflection of the first asymptotic across the real line. The sizes of the angles are conserved, so the trigonometry is the same. We know that angle ACE = Π/2 because it subtends the same chord as angle Ap'E, where p' = $M_a(p)$ = 0.

One version of the half-angle formula for the tangent is

$$\tan(\Pi/2) = \frac{\sin(\Pi)}{1+\cos(\Pi)}$$

In Figure 3, we let p' = $M_a(p)$.

$$\sin(\Pi) = \frac{[\mathrm{AE}]}{[p'\,E]}, \quad \cos(\Pi) = \frac{1}{[p'\,E]} \text{ and } 1 + \cos(\Pi) = \frac{[p'\,E]+1}{[p'\,E]}$$

Then

$$\tan(\Pi/2) = \frac{\frac{[\mathrm{AE}]}{[p'\,E]}}{\frac{[p'\,E]+1}{[p'\,E]}} = \frac{[\mathrm{AE}]}{[p'\,E]+1} \qquad (1)$$

Hence, we must find [AE] and [p'E].

Let q' = $M_a(q)$ and $\tilde{q}' = \mathcal{I}_{\mathrm{PD}}(q') = \frac{1}{\bar{q}'}$. Then the radius AE of the circle centred at E is

$$[\mathrm{AE}] = \frac{|\tilde{q}'-q'|}{2} = \frac{|\tilde{q}'-q'|}{2},$$

which Vasco converts to $\frac{1-|q|^2}{2|q|}$:

$$\frac{|\tilde{q}'-q'|}{2} = \frac{|1/\bar{q}'-q'|}{2} = \frac{1-|q'|^2}{2|q'|}, \text{ “since } |q'| \leq 1\text{”}$$

To find [p'E],

$$[p'\,E]^2 = [\mathrm{AE}]^2 + [0\,A]^2 = [\mathrm{AE}]^2 + 1 = \left(\frac{1-|q'|^2}{2|q'|}\right)^2 + 1$$

$$= \frac{1-2|q'|^2+|q'|^4+4|q'|^2}{4|q'|^2} = \frac{1+2|q'|^2+|q'|^4}{4|q'|^2} = \left(\frac{1+|q'|^2}{2|q'|}\right)^2$$

$$[p'E] = \frac{1+|q'|^2}{2|q'|}$$

Substitute [AE] and [p'E] into (1)

$$\tan(\Pi/2) = \frac{[AE]}{[p'E]+1} = \frac{\frac{1-|q'|^2}{2|q'|}}{\frac{1+|q'|^2}{2|q'|}+1} = \frac{1-|q'|^2}{1+2|q'|+|q'|^2} = \frac{1-|q'|}{1+|q'|} =$$

On p. 319, we find the equation H{0,z} = $\ln(\frac{1+|z|}{1-|z|})$, so

$\ln(\frac{1-|q'|}{1+|q'|})$ = - H{0,q'} = -D (the distance from p' to q')

ln(tan(Π/2)) = -D

tan(Π/2) = e^{-D}

which verifies the Bolyai-Lobachevsky equation for the angle of parallism in the Poincaré disc.