Needham, Chapter 6, Exercise 17, p. 335.
G. Palmer, January 11, 2017, 8:30 pm, PST

17 Suppose we have a conformal map of a surface in the xy-plane, with the metric given by (15):

$$d\hat{s} = \Lambda(x, y)\sqrt{dx^2 + dy^2}\,.$$

An elegant result from differential geometry states that the Gaussian curvature at any point on the surface is given by

$$k = -\frac{1}{\Lambda^2}\,\Delta\,(\ln \Lambda),$$

where $\Delta \equiv (\partial_x{}^2 + \partial_y{}^2)$ is the Laplacian. Try this out on the metric (16) of the stereographic map of the sphere, and on the metrics (31) and (44) of the half-plane and disc models of the hyperbolic plane.

The sphere

$$d\hat{s} = \frac{2}{1+|z|^2}\,ds \tag{16}$$

$$\frac{2}{1+|z|^2}\,ds = \frac{2}{1+x^2+y^2}\sqrt{dx^2+dy^2}\,, \quad \text{because } |z|^2 = x^2+y^2$$

We deduce the expansion factor Λ and substitute into the equation for k.

$$\Lambda = \frac{2}{1+x^2+y^2}$$

$$k = \frac{-1}{\left(\frac{2}{1+x^2+y^2}\right)^2}(\partial_x{}^2+\partial_y{}^2)(\ln(\frac{2}{1+x^2+y^2}))$$

$$= \frac{-(1+x^2+y^2)^2}{4}[\partial_x{}^2\ln(\frac{2}{1+x^2+y^2}) + \partial_y{}^2\ln\left(\frac{2}{1+x^2+y^2}\right)]$$

Find $\Delta(\ln \Lambda)$ by finding $(\partial_x{}^2(\ln \Lambda) + \partial_y{}^2(\ln \Lambda))$

$$\partial_x\ln(\frac{2}{1+x^2+y^2}) = -\frac{2x}{1+x^2+y^2}$$

$$\partial_x{}^2(\ln(\frac{2}{1+x^2+y^2})) = \frac{4x^2}{(1+x^2+y^2)^2} - \frac{2}{1+x^2+y^2}$$

$$= \frac{4x^2 - 2(1+x^2+y^2)}{(1+x^2+y^2)^2}$$

$$\partial_y{}^2(\ln(\frac{2}{1+x^2+y^2})) = \frac{4y^2}{(1+x^2+y^2)^2} - \frac{2}{1+x^2+y^2}$$

$$= \frac{4y^2 - 2(1+x^2+y^2)}{(1+x^2+y^2)^2}$$

$$(\partial_x{}^2+\partial_y{}^2)(\ln(\frac{2}{1+x^2+y^2})) = \frac{4x^2 - 2(1+x^2+y^2) + 4y^2 - 2(1+x^2+y^2)}{(1+x^2+y^2)^2}$$

$$= \frac{4(x^2+y^2)}{(1+x^2+y^2)^2} - \frac{4}{1+x^2+y^2}$$

$$= \frac{4[(x^2+y^2) - (1+x^2+y^2)]}{(1+x^2+y^2)^2} = \frac{4[-1]}{(1+x^2+y^2)^2}$$

$$= -\frac{4}{(1+x^2+y^2)^2}$$

$$k = \frac{-(1+x^2+y^2)^2}{4}(\partial_x{}^2+\partial_y{}^2)(\ln(\frac{2}{1+x^2+y^2})) = \frac{-(1+x^2+y^2)^2}{4}(-\frac{4}{(1+|z|^2)^2})$$

$$= 1$$

This is the correct value for the curvature of the sphere.

The half-plane

$$d\hat{s} = \frac{ds}{y} = \frac{\sqrt{dx^2+dy^2}}{y} \tag{31}$$

$$\Lambda = \frac{1}{y}$$

$$k = -\frac{1}{(1/y)^2}\Delta(\ln\frac{1}{y})$$

$$\partial_x{}^2\left(\ln\frac{1}{y}\right) = 0$$

$$\partial_y{}^2\left(\ln\frac{1}{y}\right) = \frac{1}{y^2}$$

$$k = -y^2/y^2 = -1$$

k = -1 is correct for the hyperbolic plane.

The disc model

$$d\hat{s} = \frac{2}{1-|z|^2}\, ds \tag{44}$$

$$\Lambda = \frac{2}{1-|z|^2} = \frac{2}{1-x^2-y^2}$$

$$\Delta(\ln \Lambda) = (\partial_x{}^2+\partial_y{}^2)(\ln(\frac{2}{1+x^2+y^2}))$$

$$= \frac{4}{\left(x^2+y^2-1\right)^2}$$

$$k = -\frac{1}{\Lambda^2}\,\Delta\,(ln\,\Lambda\,) = -\,\frac{1}{\left(\frac{2}{1-x^2-y^2}\right)^2}\,\frac{4}{\left(x^2+y^2-1\right)^2}$$

$$= -\frac{\left(1-x^2-y^2\right)^2}{4}\,\frac{4}{\left(x^2+y^2-1\right)^2} = -\frac{\left(-\left(1-x^2-y^2\right)\right)^2}{4}\,\frac{4}{\left(x^2+y^2-1\right)^2} = -1$$

We again get k = -1 for the disc model of the hyperbolic plane.