Needham, Chapter 6, Exercise 14, p. 333.
G. Palmer, December 29, 2016, 2:50 pm PST

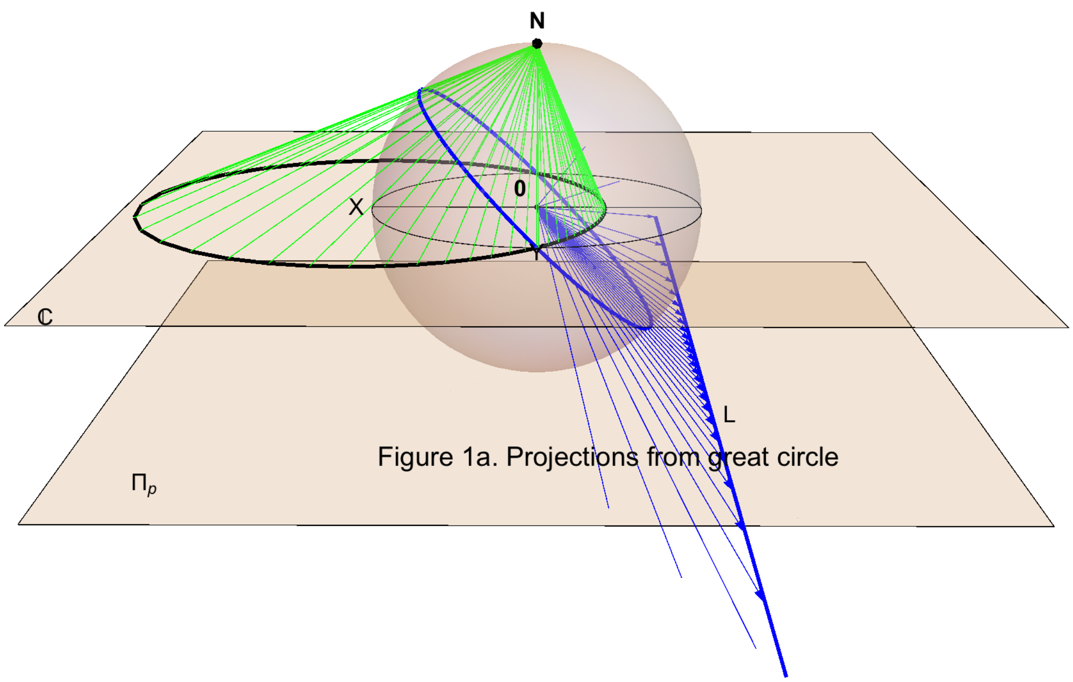

Figure 1a. Projections from great circle

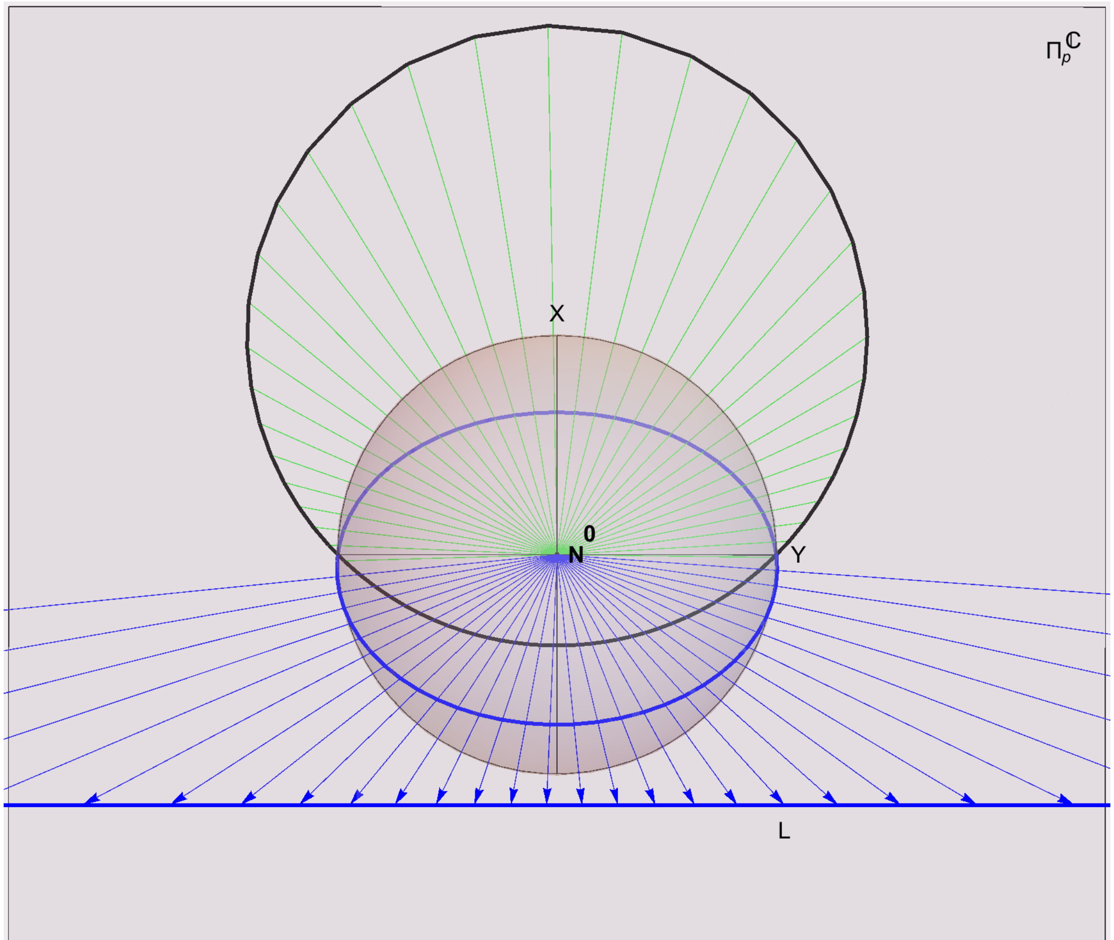

Figure 1b. Top view, projections from great circle

Out[134]=

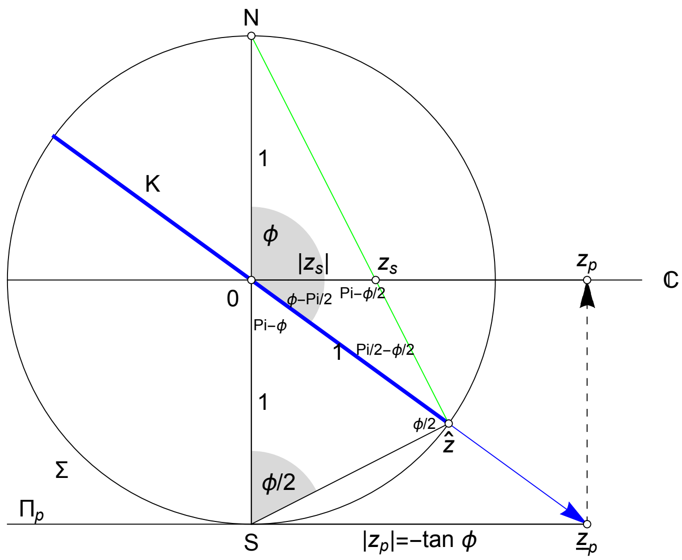

Figure 2. Collapse to ℂ

14 (i) The figure above [problem 14, p. 333] *superimposes the stereographic and projective [see p. 284] images of a great circle on the unit sphere. Let z_s and z_p be the stereographic and projective images of the point whose spherical-polar coordinates are (ϕ, θ). Referring to (21),p. 147, $z_s = \cot(\phi/2)\, e^{i\theta}$. Show that $z_p = [-\tan \phi]\, e^{i\theta}$, and deduce that*

$$z_p = \frac{2 z_s}{1 - |z_s|^2}.$$

Compare this with Ex. 9!

The figure above problem 14, suggests that a polar stereographic projection has taken an arbitrary great circle K to a circle in ℂ. The arrows appear to illustrate central projections (blue arrows) onto a line L on a plane Π_p, which is perhaps parallel to the plane of the unitcircle. In Vasco's view (Forum, Dec 28, 2016), the reference to p. 284 makes it certain that Π_p is parallel to ℂ.

But the stereographic projection comes from a pole, say N, so the dotted line 0 to z_s in the figure must represent the shadow on ℂ of the line of s-projection from N to z_s in ℂ seen from directly above and superimposed on a line of c-projection from 0 to z_p in Π. See Figures 1a and 1b. Vasco wrote "Needham's diagram on page 333 should be thought of as a diagram of the complex plane after the unit circle, the stereographic projection of the great circle, and the projective line L, have all been superimposed (collapsed down) onto the unit circle." This is an useful perspective and I have incorporated it into Figure 3 with the dashed arrows to show the points that are projected to ℂ.

Figure 2 is a cross section of Σ seen from a position along an axis perpendicular to a plane perpendicular to ℂ with angle θ measured from the x axis on the xy plane. The s-projection to z_s from N (green) passes through the disc of the unit circle and intersects K (blue). By this construction, we can see that the angle $S0z_p$ has a tangent that can be calculated as follows:

$$\tan(\mathrm{Pi}-\phi) = \frac{\sin(\mathrm{Pi}-\phi)}{\cos(\mathrm{Pi}-\phi)} = \frac{\sin(\phi)}{-\cos(\phi)} = -\tan(\phi)$$

This gives $|z_p|$. Multiply by $e^{i\theta}$ to place z_p at the proper angle with respect to the plane of projection: $z_p = [-\tan(\phi)]\, e^{i\theta}$.

...and deduce that

$$z_p = \frac{2 z_s}{1-|z_s|^2}.$$

Solution

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = (2 \sin\phi/2 \cos\phi/2)\big/\left(\cos^2\phi/2 - \sin^2\phi/2\right) \qquad (1)$$

Apply the law of sines to triangle $\hat{z}0z_s$ to get an expression in $|z_s|$ that can be substituted into (1).

$$\frac{|z_s|}{\sin(\mathrm{Pi}/2-\phi/2)} = \frac{1}{\sin(\mathrm{Pi}-\phi/2)} \Longrightarrow \frac{|z_s|}{\cos(\phi/2)} = \frac{1}{\sin(\phi/2)}$$

$$\sin(\phi/2) = \frac{\cos(\phi/2)}{|z_s|} \qquad (2)$$

Substitute (2) into (1).

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = (2 \sin\phi/2 \cos\phi/2)\big/\left(\cos^2\phi/2 - \sin^2\phi/2\right) = \left(2\,\frac{\cos(\phi/2)}{|z_s|} \cos\phi/2\right)\Big/\left(\cos^2\phi/2 - \left(\frac{\cos(\phi/2)}{|z_s|}\right)^2\right)$$

$$= \left(2\cos^2(\phi/2)\,|z_s|\right)\big/\left(\cos^2(\phi/2)\,|z_s|^2 - \cos^2(\phi/2)\right) = \left(2\cos^2(\phi/2)\,|z_s|\right)\big/\left(\cos^2\phi/2\,(|z_s|^2-1)\right) = (2\,|z_s|)\big/(|z_s|^2-1)$$

So

$$z_p = [-\tan\phi]\, e^{i\theta} = -\,(2\,|z_s|)\big/(|z_s|^2-1)e^{i\theta} \equiv (2 z_s)\big/(1-|z_s|^2)$$

Compare this with Ex. 9!

In both problems there is a stereographic projection. In Ex. 9, an h-line on the Poincaré disc is s-projected from S to a hemicircle on Σ. In Ex. 14, a great circle on Σ is s-projected from N to a circle in ℂ. In Ex. 9, the hemicircle resulting from the s-projection is projected vertically to a chord in the PD. In Ex. 14, a central projection takes the great circle to a line in a plane parallel to ℂ and touching Σ at the south pole. Both the vertical projection and the c-projection section the sphere, but in different ways. Both the v-projection and the c-projection take points from a circle to points in a line.

In Ex. 9, the vertical projection to the chord in ℂ is described by

$$z' = (2\,z)/\left(1 + |\,z\,|^2\right)$$

where the s-projection takes z to the semicircle on Σ, and the v-projection takes $\tilde{z}$ to z'.

In Ex. 14, s-projection takes points in the great circle directly to points in a circle on ℂ, and the c-projection takes those same points in the great circle to a line in the projective plane. The c-projection can be described by

$$z_p = (2\,z_s)/\left(1 - |\,z_s\,|^2\right)$$

where z_s in ℂ is the stereographic projection from a point in Σ. We see that the two problems have in common the fact that a point of projection to the line can be described in terms of projection from ℂ to a point of projection in Σ (Ex. 9) or from Σ to a point of projection in ℂ (Ex. 14). The two equations have similar structures.

Seen from above, in Ex. 9, the s-projection is transposed on the v-projection, and in Ex. 10, the s-projection is transposed on the c-projection. We see some similarities in the structures of Ex.s 9 and 14, but we don't see a unifying principle emerging from the comparison.

Note: Vasco has an useful perspective on the problem summed up in this quote from the Forum, December 29, 2016: *Try and get used to the idea of two complex planes; if it helps psychologically, then draw two separate diagrams, one for the stereographic projection, with the equatorial plane as the complex plane, and another for the central projection, with the plane of projection as the complex plane, and then at the end superimpose the two complex planes. I only put them on the same diagram for convenience. In my head I am thinking of them as completely separate.*