Needham, Ch. 6, Exercises (ii-vi)
G. Palmer, January 3, 2017, 12:45 pm, PST

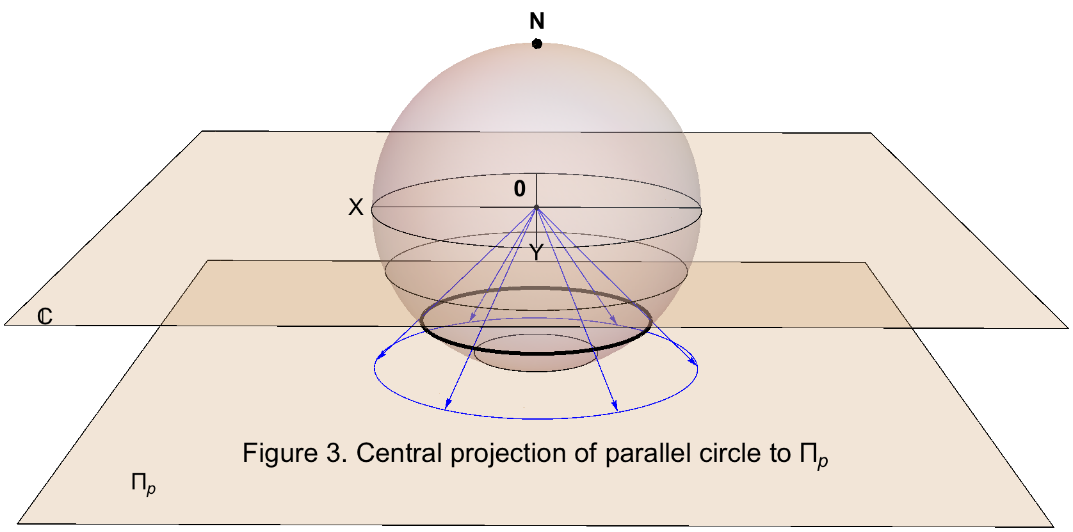

Figure 3. Central projection of parallel circle to Π_p

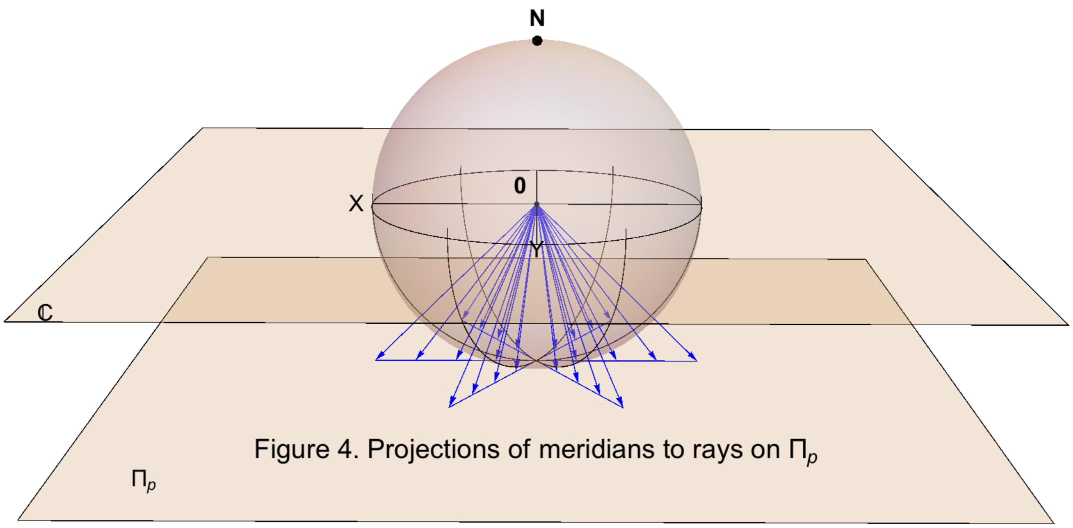

Figure 4. Projections of meridians to rays on Π_p

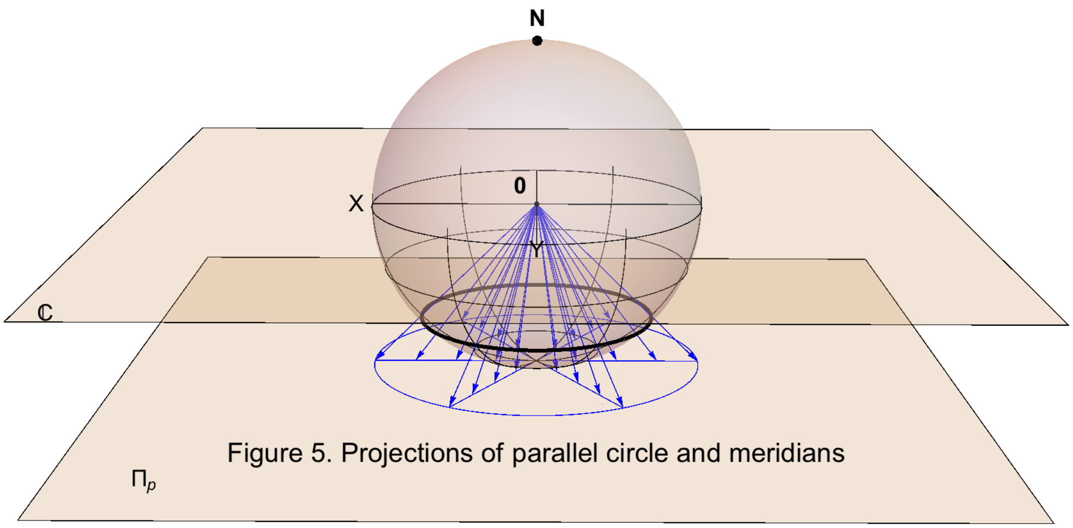

Figure 5. Projections of parallel circle and meridians

(ii) Sketch the curves on the hemisphere that are centrally projected to circles $|z_p|$ *= const. and to the rays arg* z_p *= const. Although angles are generally distorted in the projective model, observe that these circles and rays really are orthogonal, as they appear to be.*

The centrally projected curves that go to circles are elliptic transformations on Σ. We draw them as a family of circles on Σ with parallel planes orthogonal to the standard N/S axis. They are projected from N to a parallel complex plane that touches S.

The centrally projected curves that go to rays are meridians. Tangents of the meridians and parallel circles are orthogonal. When projected from the origin to the projective plane, they remain orthogonal.

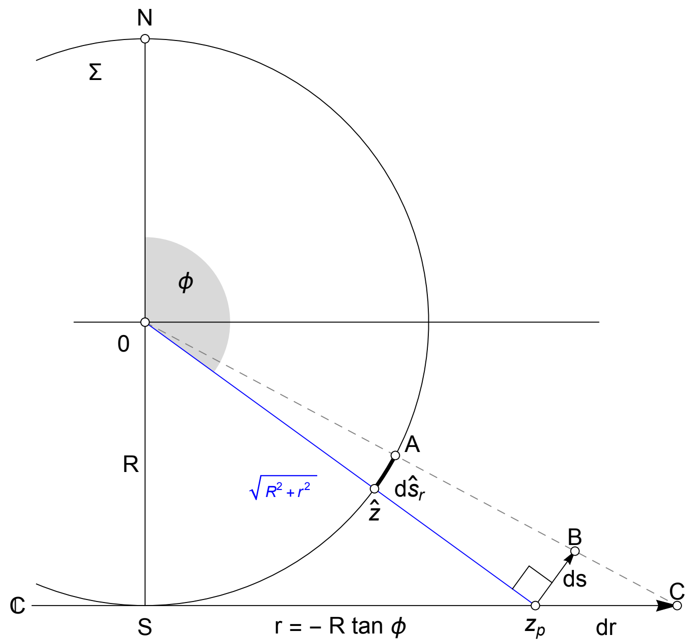

Figure 6. $d\hat{s}_r$

(iii) Now let the sphere have radius R, and write $z_p = r\, e^{i\theta}$ for the projective image of the point (ϕ, θ). Thus, $r = -R \tan\phi$. Show that if z_p moves a distance dr along the ray θ = const., then the corresponding point on the sphere moves a distance $d\,\hat{s}_r$ given by

$$d\,\hat{s}_r = \frac{dr}{1+(r/R)^2}.$$

Since θ = const., we can plot the problem in two dimensions and place the ℂ plane tangent to Σ at S. Then we can write $r = -R \tan\phi$ and $z_p = r\, e^{i\theta}$. The infinitesimal vector ds is plotted perpendicular to z_p. As dr tends to zero, the triangle z_pBC tends to similarity with the triangle S0z_p. Then

$$\frac{1}{R}\sqrt{\left(R^2+r^2\right)} = \frac{\mathrm{dr}}{\mathrm{ds}} \qquad \Longrightarrow \qquad \mathrm{ds} = (R\,\mathrm{dr})\Big/\sqrt{\left(R^2+r^2\right)}$$

and by the similar triangles $\hat{z}\,0\,A$ and $z_p\,0\,B$,

$$d\,\hat{s}_r = R\Big/\sqrt{\left(R^2+r^2\right)}\ \mathrm{ds} = R\Big/\sqrt{\left(R^2+r^2\right)}\ (R\,\mathrm{dr})\Big/\sqrt{\left(R^2+r^2\right)} = \left(R^2\,\mathrm{dr}\right)\Big/\left(R^2+r^2\right) = \mathrm{dr}\Big/\left(1+(r/R)^2\right)$$

(iv) Likewise, show that the separation $d\hat{s}_\theta$ of the points on the sphere corresponding to the points (r, θ) and (r, θ + dθ) in the map is given by

$$d\hat{s}_\theta = (r\,d\theta)\Big/\left(\sqrt{1+(r/R)^2}\right).$$

This is easier then (iii), because ds is easily calculated as r dϕ. Then by similar triangles $0\hat{z}A_\theta$ and $0\hat{z}B_\theta$ in the plane $0\hat{z}A_\theta$, where $A_\theta = R\,e^{i(\theta + \mathrm{d}\theta)}$ and $B_\theta = \sqrt{R^2 + r^2}\; e^{i(\theta + \mathrm{d}\theta)}$ centred at S,

$$\mathrm{d}\hat{s}_\theta = \left(R \middle/ \left(\sqrt{R^2 + r^2}\right)\right) \mathrm{r}\ \theta = (r\,\mathrm{d}\theta) \middle/ \left(\sqrt{1 + (r/R)^2}\right),$$

by dividing top and bottom by R and moving R under the square root sign.

(v) Deduce that the spherical separation d$\hat{s}$ corresponding to the points (r, θ) and (r + dr, θ + dθ) is given by

$$d\,\hat{s}^2 = \frac{dr^2}{\left(1+(r/R)^2\right)^2} + \frac{r^2\,d\theta^2}{1+(r/R)^2}$$

$$d\,\hat{s}^2 = \left(d\,\hat{s}_r\right)^2 + \left(d\,\hat{s}_\theta\right)^2$$

$$\left(d\,\hat{s}_r\right)^2 = \left(\mathrm{dr}/\left(1 + (r/R)^2\right)\right)^2 = \frac{dr^2}{\left(1+(r/R)^2\right)^2}$$

$$\left(d\,\hat{s}_\theta\right)^2 = \left((r\,\mathrm{d}\theta) \middle/ \left(\sqrt{1 + (r/R)^2}\right)\right)^2 = \frac{r^2\,d\theta^2}{1+(r/R)^2} \qquad \text{Done.}$$

(vi) The Beltrami metric of the hyperbolic plane (50) appears in Ex. 13 (v):

$$\left(d\,\hat{s}_\theta\right)^2 = \frac{dr^2}{\left(1+(r/R)^2\right)^2} + \frac{r^2\,d\theta^2}{1+(r/R)^2} = \frac{dr^2}{\left(1+(r/i)^2\right)^2} + \frac{r^2\,d\theta^2}{1+(r/i)^2}$$

$$= \frac{dr^2}{\left(1-r^2\right)^2} + \frac{r^2\,d\theta^2}{1-r^2} \qquad (50)$$

What happens if we rewrite this in terms of quaternions?