Needham, Chapter 6, Exercise 13, p. 332.
G. Palmer, December 14, 2016, 9 am PST

13 Let $z = r\, e^{i\theta}$ denote a point in the Klein model of the hyperbolic plane.

(i) On the hemisphere model lying above the Klein model, sketch the images under vertical upward projection of some circles r = const. and some rays θ = const. Although angles are generally distorted in the Klein model, deduce that these circles and rays are really orthogonal, as they appear to be. Also note that the Euclidean circles r = const. are also h-circles.

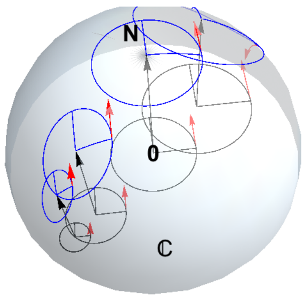

Figure 1

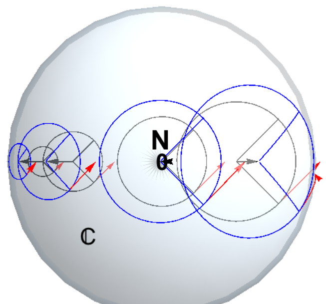

Figure 2

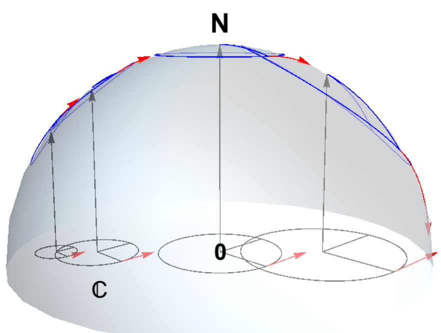

Figure 3

In the Klein model, vertical projection of the circles in ℂ evidently sends them to ellipses on the hemisphere. Rays on the Klein model circles can be viewed as segments of chords, which go to segments of vertical sections of the hemisphere. Needham does not claim that the lines on the hemisphere produced by vertical sections are h-lines, but at least the line above a diameter (Figure [b], p. 324) is one. In the ℂ plane, tangents to the circles at the end points of rays are orthogonal to the rays. These tangents are sections of chords, so they also go to vertical sections of the hemisphere. As Vasco pointed out, the tangents are parallel when drawn from the point of intersection of several concentric circles with a single ray extending to the outermost. If we choose only circles centered at the origin of the Klein model, such as the origin centred black circle in Figure 1, then all the rays are segments of h-lines. Hence, when projected vertically to the hemisphere (blue), their images are also h-lines. On the hemisphere, at the point of intersection of the rays of the ellipses with the vertically projected tangents, the angles must be orthogonal. Inspection of their full projections suggests that they remain orthogonal when produced.

I would go beyond this and propose that we can relax the constraint that requires the circles in the Klein model be centered at the origin. Noting that the Euclidean circles in the Klein model are also h-circles, and that the rays and tangents are chords corresponding to segments of h-lines also suggests that they should remain orthogonal at infinitesimal distances from the point of intersection when vertically projected to the hemisphere.

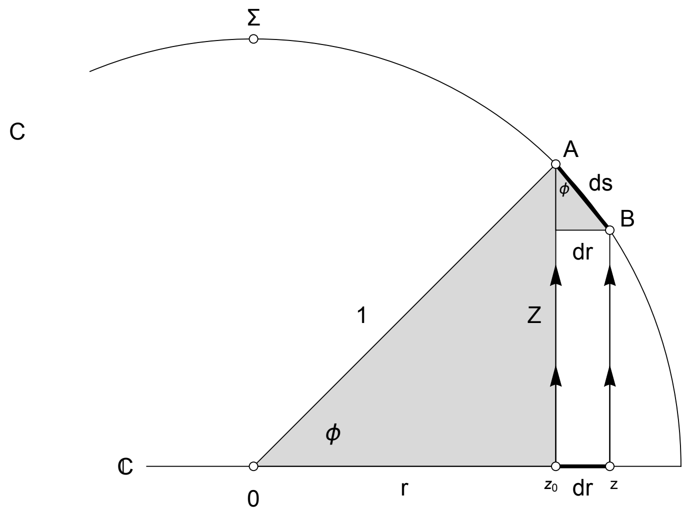

Figure 4. Facsimile of Figure [b], p. 332

(ii) Figure [b] above shows a vertical cross section of the hemisphere model and Klein model [on ℂ] taken through a ray θ = const. If the point z moves outward along this ray by dr, let ds denote the movement of its vertical projection on the hemisphere. Explain why the two shaded triangles are similar, and deduce that ds = (dr/Z).

Angle 0AB is a right angle, so angle 0 Az_0 = Pi-ϕ, angle z_0 AB = ϕ, and angle ABC = Pi-ϕ. Hence, the two triangles have three angles in common, so they are similar. By inspection, we can write

ds/1 = dr/Z

because ds and 1 are the hypotenuses and [BC] = dr and Z are both adjacent to ϕ^c (ϕ-complement), so ds = dr/Z.