Needham, Chapter 6, Problem 12, p. 331,332, Representations of orthogonal h-lines in the Klein hemisphere model.
G. Palmer, December 11, 7 pm, PST

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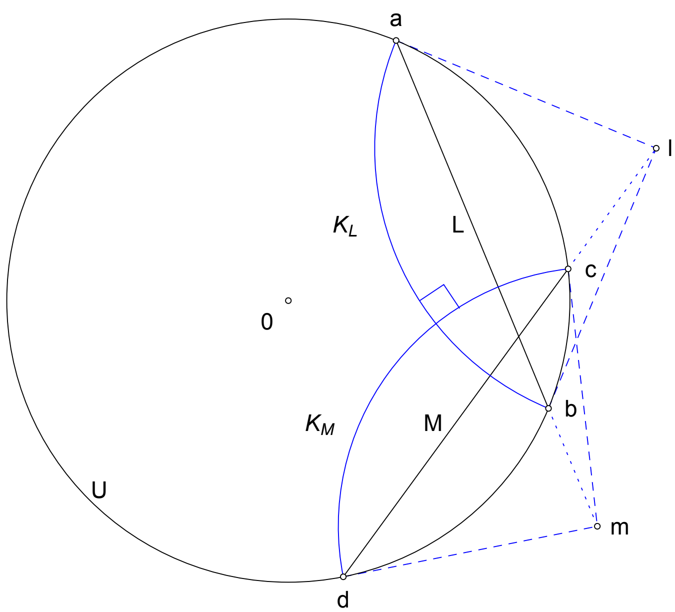

Figure 1. Ch. 6, Ex. 12,
Representations of orthogonal h−lines
in the Klein hemisphere model

12 Let L and M be two intersecting chords of the unit circle, and let l and m be the intersection points of the pairs of tangents drawn at the ends of these chords. See figure [a] below. In the Klein model, show that L and M represent orthogonal h-lines if and only if l lies on M (produced) and m lies on L (produced).

Vasco solved this problem. I provide my own view of it here. Once again, it is important to realize that the term “h-lines” refers to the h-plane. In recent problems, objects in the h-plane seem not to appear as objects in the diagrams being used to calculate or infer them. We see only *representations* of them in the PUHP, PD or Klein models. Objects in the h-plane may have different dimensions from those in the models and maps. In this problem, even though the exercise is ostensibly designed to provide instruction on the Klein hemisphere model, we need also to consider representations of h-lines that actually belong to the Poincaré disc superimposed on the ℂ plane of the Klein model. In Figure 1, representations of h-lines in the Klein model appear in black. Representations of h-lines in the PD appear in Blue.

The straight lines of the Klein model represent vertical sections of the hemisphere, and *h-lines in the Klein-model are straight Euclidean chords of the unit circle* (p. 323). Then we would want the angle between two h-lines to be Pi/2, but the Klein model is non-conformal, so we can't expect that orthogonal chords will project to orthogonal h-lines. But in any case, we don't see orthogonal chords in Needham's Figure [a].* To solve this problem, one needs the following theorem from Needham, p. 129.

The inverse $\tilde{z}$ of z in K is the second intersection point of any two circles that pass through z and are orthogonal to K.

To begin, we take K_L and K_M to be the arcs of circles centered at l (small L) and m, as found by the intersection of the tangents from the ends of the chords L and M as in Figure 1. We can say immediately that these arcs in the PD *correspond* to L and M, which are the straight line *representations* of h-lines in the Klein model. If the arcs can be shown to be orthogonal to the unit circle U, then they *represent* h-lines in the PD. Let a and b be the intersection points of K_L with U, and let c and d be the intersection points of K_M with U. It is given in Needham's figure [a] that K_L is orthogonal to K_M. If in fact a = $\tilde{b}$, an inversion in K_M, then by the theorem, U must be orthogonal to K_M, and similarly, if c = $\tilde{d}$, U must be orthogonal to K_L. Then the line joining the points of intersection of two circles must, when produced, pass through the center of the circle of inversion. So m must lie in L produced and l must lie in M produced.

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*However, in exercise 13 (i) we read that Euclidean circles on the unit circle also project as h-lines on the hemisphere. Needham says, "although angles are generally distorted in the Klein model, deduce that these circles and rays really are orthogonal, as they appear to be." Then it seems, orthogonal circles and rays on the unit circle actually can be projected to orthogonal circles and rays on the hemisphere. We hope to find out what Needham meant when we do problem 13.