Neeham, Chapter 6, Exercise 11, p. 331.
G. Palmer, December 12, 2016, 10:10 am, PST

11 (i) In the hyperbolic plane, show that the h-circumference of an h-circle of h-radius ρ is 2 Pi sinh ρ. [Hint: Represent the h-circle as an origin-centred Euclidean circle in the Poincaré disc.]

Original Solution 1 has been deleted.

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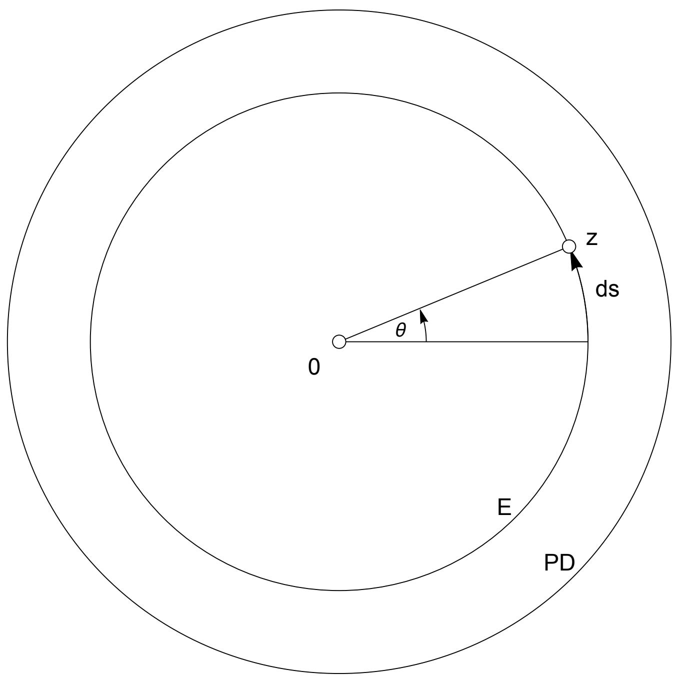

Figure 1. Ch. 6, Ex. 11, h−circumference of h−circle E, radius ρ, on PD

Solution using the metric of the Poincaré disc

Figure 1 shows the situation in the Poincaré disc.

$$d\hat{s} = \frac{2}{1-|z|^2}\, ds \quad (44,\ \text{p. } 318)$$

This is a strange construction: $d\hat{s}$ is an infinitesimal arc length in the h-plane, which is not shown. The infinitesimal ds is in the PD, and |z| is Euclidean. In the figure, it appears that ρ = |z|, but that is not the case. Both ρ and the h-circumference are offstage. One could draw z anywhere on the circle E (for Euclidean).

The h-circumference h-Cf is the integral of h-arc length $d\hat{s}$ over the angle 2Pi.

$$\text{h-Cf} = \int_0^{2\,\text{Pi}} d\hat{s} = \int_0^{2\,\text{Pi}} \frac{2}{1-|z|^2}\, ds$$

$$= \int_0^{2\,\text{Pi}} \frac{2}{1-|z|^2}\,(|z|\, d\theta) = \int_0^{2\,\text{Pi}} \frac{2|z|}{1-|z|^2}\, d\theta$$

$$= 2 \left|_0^{2\,\text{Pi}} \frac{|z|}{1-|z|^2}\theta = \frac{4\,\text{Pi}\,|z|}{1-|z|^2} \qquad (1)\right.$$

We want *2 Pi sinh* ρ, so we need |z| in terms of ρ.

$$\rho = \text{H}\{0,z\} = \ln \frac{1+|z|}{1-|z|}$$

Take the exponential of both sides and solve for |z|.

$$e^{\rho} = \frac{1+|z|}{1-|z|}$$

$$(1-|z|)\, e^{\rho} = 1 + |z|$$

$$e^{\rho} - |z|\, e^{\rho} = 1 + |z|$$

$$|z| + |z|\, e^{\rho} = e^{\rho} - 1$$

$$|z| = \frac{e^{\rho}-1}{e^{\rho}+1} \qquad (2)$$

Calculate the denominator of the circumference in equation (1).

$$1-(|z|)^2 = 1 - \left(\frac{e^{\rho}-1}{e^{\rho}+1}\right)^2$$

$$= \frac{(e^{\rho}+1)^2}{(e^{\rho}+1)^2} - \left(\frac{e^{\rho}-1}{e^{\rho}+1}\right)^2 = \frac{(e^{\rho}+1)^2-(e^{\rho}-1)^2}{(e^{\rho}+1)^2} \qquad (3)$$

Substitute (2) and (3) into (1).

$$\frac{|z|}{1-|z|^2} = \left(\frac{e^{\rho}-1}{e^{\rho}+1}\right)\frac{(e^{\rho}+1)^2}{(e^{\rho}+1)^2-(e^{\rho}-1)^2} = \frac{(e^{\rho}-1)(e^{\rho}+1)}{(e^{\rho}+1)^2-(e^{\rho}-1)^2}$$

$$= \frac{e^{2\rho}-1}{e^{2\rho}+2\,e^{\rho}+1-(e^{2\rho}-2\,e^{\rho}+1)} = \frac{e^{2\rho}-1}{4\,e^{\rho}}$$

$$= \frac{e^{\rho}-e^{-\rho}}{4} = \frac{\sinh(\rho)}{2}$$

$$\text{h-Cf.} = \frac{4\,\text{Pi}\,|z|}{1-|z|^2} = \frac{4\,\text{Pi}\sinh(\rho)}{2} = 2\,\pi\sinh(\rho)$$

This is what we wanted. In Figure 1, we let the radius of the PD equal 1. The Euclidean disc has radius

.75. θ = Pi/8. Calculations give

$\lvert z\rvert$ = 0.75	(Euclidean metric)
ρ = 1.94591	(ln[(1+$\lvert z\rvert$)/(1-$\lvert z\rvert$)])
ds = 0.294524	(Euclidean metric)
d$\hat{s}$ = 1.3464	(PD metric)
E-Cf = 4.71239	(Euclidean distance)
h-Cf = 21.5423	(2 Pi Sinh[ρ])

ds/d$\hat{s}$ = E-Cf/h-Cf = 0.21875

h-Cf/ρ = 11.070

This shows that while ρ appears to equal r in the figure, it is in fact over twice as large. d$\hat{s}$ in Σ-space is over four times as large as ds in the PD. As would be expected, ds/d$\hat{s}$ = E-Cf/h-Cf . Even more interesting is the ratio of h-circumference to the h-radius ρ: h-Cf/ρ = 11.070, which is nearly double the Euclidean ratio 2π = 6.28, but the ratio varies with the h-radius ρ. For example, the ratio for ρ = .5 is 7.6256, which is closer to Euclidean. The formula is h-circumference = 2 π sinh(ρ).

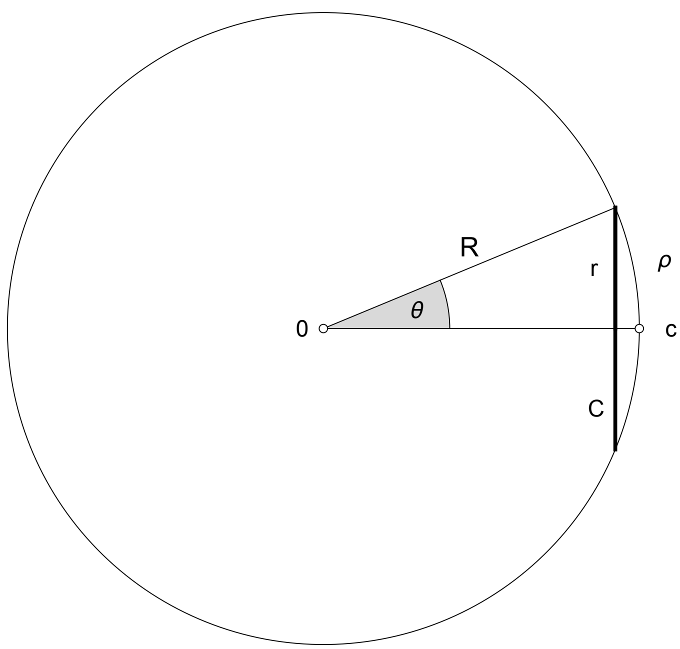

Figure 1. Ch. 6, Ex. 11, Circumference of circle C on sphere Σ

(ii) Let the inhabitants of the sphere of radius R draw a circle of (intrinsic) radius ρ. Use elementary geometry to show that the circle's circumference is 2Pi R sin(ρ/R). Show that if we take the radius of the sphere to be R = i, then this becomes the formula in part (i). [Compare this with Ex. 14.]

Figure 1 presents a view of a great circle on the sphere at a viewing angle perpendicular to the plane of the great circle. The circle C is perpendicular to Π and seen as though the viewer were in the plane of C. The intrinsic radius is any great circle segment of Σ from c (intrinsic center of C) to an intersection with C.

$\rho = R\ \theta$ so $\theta = \rho/R$

$r = R \sin(\rho/R)$

Then the circumference of C is $1\pi r = 2\ \pi\ R \sin(\rho/R)$.

If we let R = i,

$2\ \pi\ R \sin(\rho/R) = 2\ \pi\ i \sin(\rho/i)$

$= 2\ \pi\ i \sin(-i\rho) = 2\ \pi\ (-i) \sin(i\ \rho) = 2\ \pi\ (-i)\ i \sinh \rho$

$= 2\ \pi \sin h\ \rho$

I don’t see the comparison with Ex. 14, so I will return to it after completing 14. Furthermore, why would one say that R = i? Should R not always be a real number?