Needham, Chapter 6, Exercise 10, p. 331
G. Palmer, December 5, 2016, 8:10 pm, PST

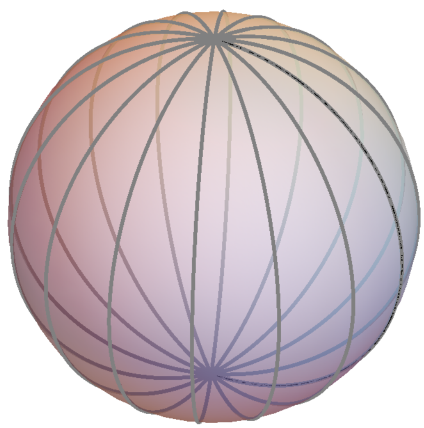

Figure 1. Sphere generated by semicircle

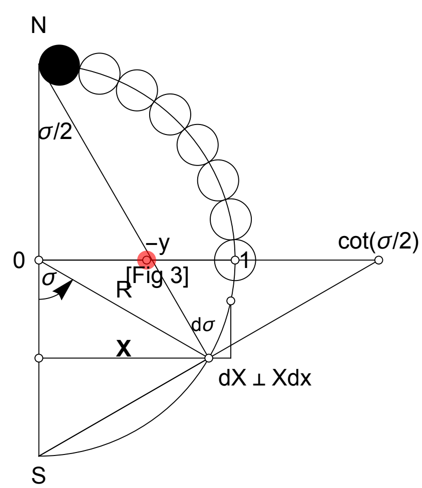

Figure 2. Semicircle generator for sphere

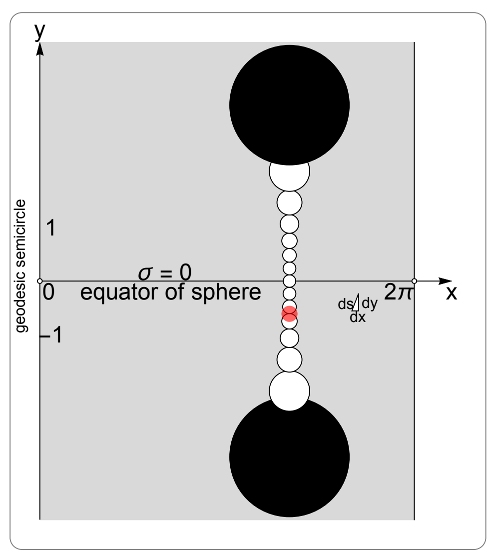

Figure 3. Map for sphere

10 Think of the sphere as the surface of revolution generated by a semicircle. Construct a conformal map of the sphere by strict analogy with the construction of the map of the pseudosphere in [20]. Show *that this is the Mercator map that you obtained in Ex. 14, p. 259.*

The Mercator map is obtained by applying ln(z) to the stereographic map. If we can show that y = -ln(cot(σ/2)), then we have shown that this is the Mercator map. See pp. 146-147 for the derivation of the 3D stereographic projection $z = \cot(\phi/2)\, e^{i\theta}$.

We follow the progression beginning on p. 296.

The radius of the sphere Σ is r = 1 and k = 1 (Figure 1). We need an x coordinate that measures angle around the axis of the sphere, restricted to $0 \le x < 2\pi$. We need a coordinate σ that measures arc length along each semiscircular generator as in [18a]. We can take σ to be the ϕ angle measured in the vertical plane passing through the origin with $\sigma = 0$ on the equator with the restriction -Pi/2 ≤ σ ≤ Pi/2. The curves x = const. are the meridians and hemispherical generators of the sphere. As with the tractrix, the semicircles are geodesics. The curves σ = const. are circular cross sections of the sphere. Only the equator is a geodesic. The radius of such a circle is the X-coordinate given by X = sin(σ). This is clear from the diagram, but it can also be derived in the same was as was done for the pseudosphere.

$$Z^2 = 1 - X^2 \quad \text{from the Pythagorean theorem}$$

$$\frac{dX}{d\sigma} = \frac{\sqrt{1-X^2}}{1} = \sqrt{1-X^2} \qquad \text{from similar triangles}$$

$$\frac{dX}{\sqrt{1-X^2}} = d\sigma$$

$$\int \frac{1}{\sqrt{1-X^2}}\, dX = \int d\sigma$$

$$\arcsin(X) = \sigma$$

$$X = \sin(\sigma)$$

In the map of the sphere, we choose the angle x as the horizontal axis, where 0 ≤ x < 2Pi for this problem. The lines x = const. must be orthogonal to images of the circles σ = const. as they are on Σ. The image of σ = const. must be represented by horizontal lines y = const., so y = y(σ), as in [20b]. From Figure 2 and similar triangles, we can see that

$$\frac{dy}{d\sigma} = \frac{dx}{X\,dx} = \frac{1}{X}$$

Then

$$dy = \frac{d\sigma}{X} = \frac{d\sigma}{\sin(\sigma)}$$

$$\int dy = \int (1/\sin(\sigma)\, d\sigma$$

$$\Longrightarrow y = -\ln(\cot(\sigma/2)) \qquad \text{(Cf. pink dots, figures 2 and 3)}$$

I consulted two tables of integrals and found no instance of $\int(1/\sin(x)\,dx$. Integrating with *Mathematica* gives y = -ln(cos(σ/2)) + ln(sin(σ/2)). This is equivalent to -ln(cot(σ/2)), showing that this is a Mercator map. Furthermore, the very fact that the map must be conformal suggests that the map is a stereographic projection of the sphere and that dy is orthogonal to dx. So one can plot (x, y) as the projection of points on the sphere onto the map.

Like all Mercator maps, objects expand as a function of their distance from the equator. The discs in Figure 3 are plotted at height y = -ln(cot(σ/2), with σ plotted from -Pi/2 to Pi/2 and σ/2 measured at the N pole. The discs in the x = const. line in Figure 3 do not appear to be conformal with a series of touching disks aligned on a meridian. I'm not sure why.

Since the map is conformal, an infinitesimal line-segment emanating from (x, σ) in any direction on the on the sphere must be multiplied by the factor (1/X) = 1/sin(σ) on the map, and the metric is d$\hat{s}$ = X ds. Consequently disc radii in the map are calculated as (dσ/2)(1/X), where dσ is the diameter of a disc on the sphere. To put it otherwise, each step increase in σ equals 2 r, where r is the radius of a disc on the sphere.