Figures pertaining to Figure [32], Needham, p. 313.
G. Palmer, November 3, 2016, 7:15 pm PST

This document reproduces a version of Figure [32] and adds two new plots illustrating the reflections composing the direct motion depicted in [32].

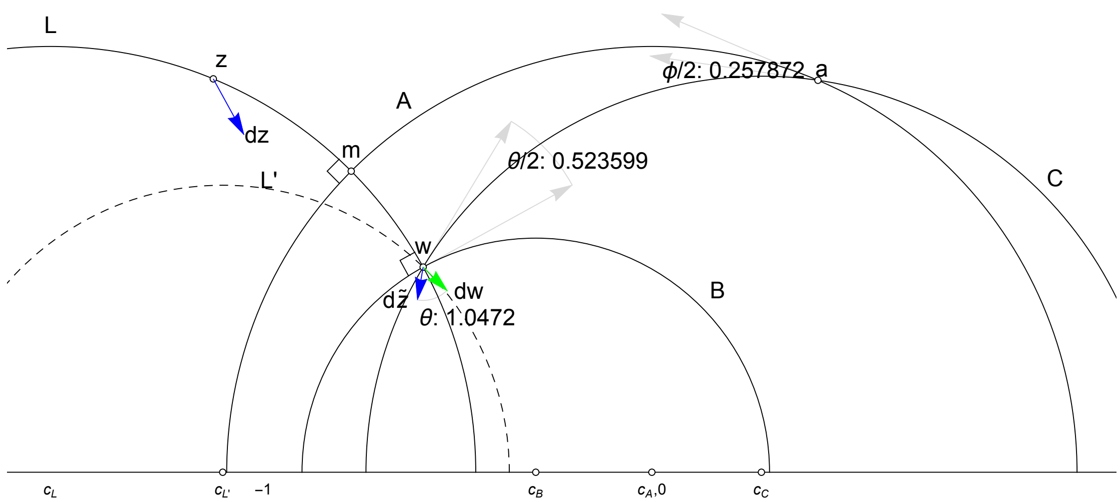

Figure 1. Facsimile of Figure [32], p. 313, M = $\mathcal{R}_w^\theta \circ \mathcal{T}_L^\delta$

In Figure 1, we attempted to follow the principles on pp. 311-313 of subsection *8 Decomposing an Arbitrary Direct Motion into Two Reflections*. On p. 312, ℙ 3, Needham says “First carry z to w using the h-translation $\mathcal{T}_L^\delta$, where δ = H{z, w}, and L is the unique h-line from z to w.” H-line L was plotted with center at -1.5 and radius 1. H-line A was plotted orthogonal to L. Point m was plotted on L at the arbitrary point Exp(Pi/4) + c_L, where c_L is the center of L. Point z was plotted as an arbitrary rotation in c_L of $e^{\mathrm{iPi/8}}$m. Point w was plotted as $\mathcal{I}_A(z)$. This accomplishes the first instruction. Then w = $\mathcal{T}_L^\delta$(z) = $M_{\mathcal{T}}$(z) = $e^{i\lambda}(z - c_L) + c_L$ is a simple rotation of z on L that takes z to w, where λ is the distance zw on L in radians. We will shortly see that this $M_{\mathcal{T}}$(z) = ($\mathcal{R}_B \circ \mathcal{R}_A$). Please note that this M is not the same as the $\mathcal{M}$ that Needham discusses in this paragraph.

Needham indicates that dz can be any point, no matter how close to z. Just because it seems about right for the diagram, we chose to apply the rotation on L to dz (prior to the rotation of z). Now we rotate (z+dz) to obtain d$\tilde{z}$ = w + M’(z) dz = w + $e^{i\lambda}$dz. (So d$\tilde{z}$ = $e^{i2\lambda}$ tan(z) in L.) Point d$\tilde{z}$ + w was actually plotted using

$d\tilde{z}$ = |dz| (Im(w)/Im(z)) i $(w - c_L)/\,|\, w - c_L \,|\, e^{i\lambda}$,

The ratio Im(w)/Im(z) was applied following [22b], p. 302 and paragraph 1, p. 303.

In the same paragraph, Needham says “Since $\mathcal{M}$ must be conformal, it can be thought of as an analytic function, so we may write dw = $\mathcal{M}$’(z)dz”, but we don’t have this $\mathcal{M}$ yet, so we have to find another way to obtain dw. We do it by working backwards, plotting B and C with an arbitrary angle θ at w, and then finding dw, which will make the angle θ with $d\tilde{z}$ about w.

B was plotted as an h-line orthogonal to L at w. An arbitrary angle of θ was chosen for the angle from $d\tilde{z}$ to dw. A tangent to C at w was found by adding $\theta/2$ to the tangent to B at w with θ = Pi/3. This means that the combination of reflections in B and C compose a single rotation of θ, as in the rotation $d\tilde{z}$ to dw in w. (Similarly, a combination of reflections A and C composes a rotation of ϕ in a, and a combination of reflections in A and B composes a rotation of λ in $c_{L.}$) With a tangent for C, one can solve for the center and plot C at c_L with a radius of $(w - c_C)$. Then dw can be plotted with $(\mathcal{R}_C \circ \mathcal{R}_B)(w+d\tilde{z})$. This is a Möbius transformation, say $M_{\text{CB}} = (\mathcal{R}_C \circ \mathcal{R}_B)$, but it is still not the $\mathcal{M}$ that we seek. We apply M_{CB} to $(\mathcal{R}_B \circ \mathcal{R}_A)$ to get $\mathcal{M} = (\mathcal{R}_C \circ \mathcal{R}_B)\ (\mathcal{R}_B \circ \mathcal{R}_A)$, which is a Möbius transformation that takes z to w AND z+dz to w + dw. We could also write this transformation as $\mathcal{M} = M_{\text{CB}} \circ M_{\mathcal{T}}$.

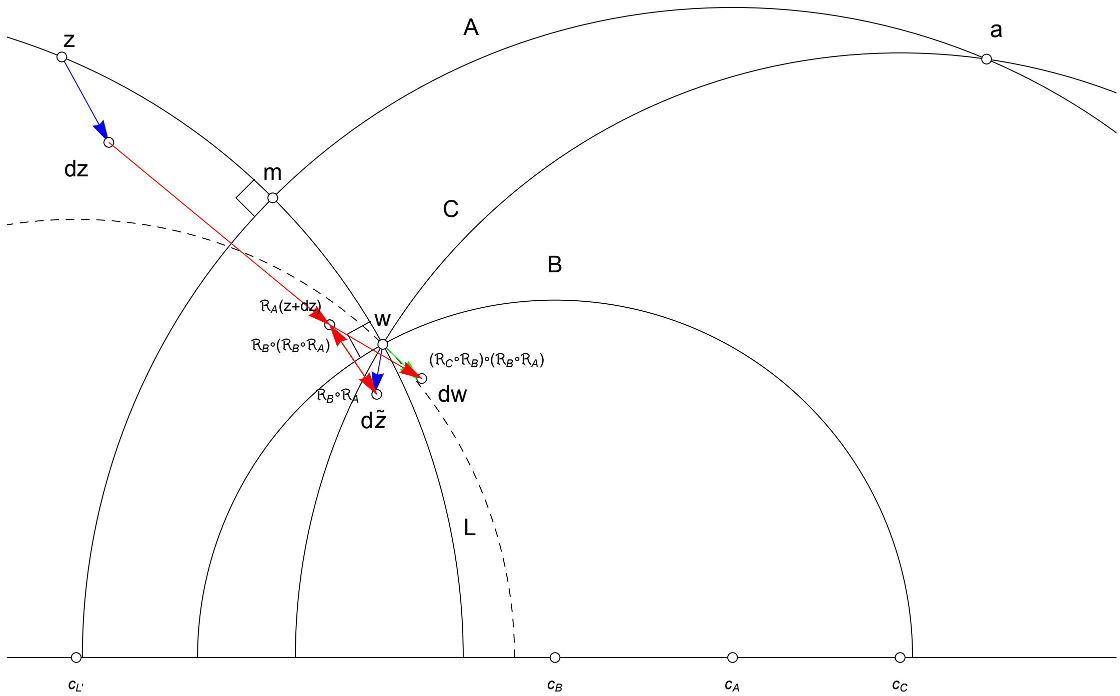

Figure 2. Reflections: M = ($\mathcal{R}_C \circ \mathcal{R}_B$) ∘ ($\mathcal{R}_B \circ \mathcal{R}_A$)

Figure 2 illustrates that the four h-reflections M = ($\mathcal{R}_C \circ \mathcal{R}_B$) ∘ ($\mathcal{R}_B \circ \mathcal{R}_A$) are equivalent to the direct motion calculated by h-translation and reflection M = $\mathcal{R}_w^\theta \circ \mathcal{T}_L^\delta$. The equation used for the reflections is from (4), p. 125:

$$\mathcal{I}_K(z) = \frac{R^2}{\text{Conjugate}[z-q]} + q$$

The following points were calculated: $\mathcal{I}_A(z + \mathrm{dz})$, $\mathcal{I}_B \circ \mathcal{I}_A(z + \mathrm{dz})$, $\mathcal{I}_B \circ \mathcal{I}_B \circ \mathcal{I}_A(z + \mathrm{dz})$,
$\mathcal{I}_C \circ \mathcal{I}_B \circ \mathcal{I}_B \circ \mathcal{I}_A(z + \mathrm{dz})$

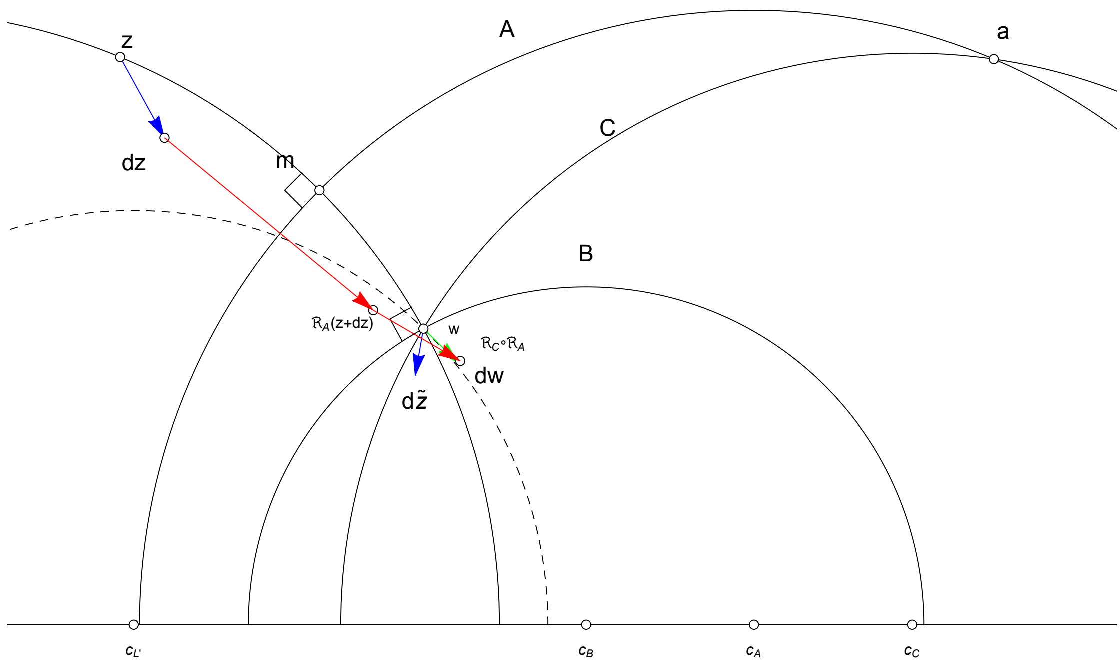

Figure 3. Decomposition of M into two h–Reflections: M = $\mathcal{R}_C \circ \mathcal{R}_A$

Figure 3 illustrates the decomposition of M into two reflections: M = $\mathcal{R}_C \circ \mathcal{R}_A$, which is possible because the two instances of $\mathcal{R}_B$ cancel one another.