Needham, Chapter 3
G.Palmer, October 24, 2016, 9 am PST

Notes on transformations in the hyperbolic plane

This document presents some related plots of transformations in the complex and hyperbolic planes as discussed on pp. 162-169 and pp. 306-311. In the case of h-rotation, limit-rotation, and limit-translation it appears that they can be transposed onto the complex plane using the inverse Möbius transformation F^{-1}(w), which replaces the fixed points. The types of movements in the two systems appear to be different. In the complex plane, the movements illustrated are elliptical, parabolic, and hyperbolic Möbius transformations (other than F^{-1}(w)). In the hyperbolic plane, the movements illustrated are h-rotation, limit-rotation, and limit-translation. The character of the hyperbolic plane is evidently different from that of the hyperbolic transformation illustrated on p. 153, as indicated in footnote 12, p. 308. I should emphasize that I am not claiming that Möbius transformations with fixed points are mathematically related to the hyperbolic plane, even though I think they must be, somehow. I am only claiming that my plots of the hyperbolic movements along hyperbolic lines and parallels can be neatly transposed onto the complex system with fixed points transformed by F^{-1}(z).

h-Rotation

Out[991]=

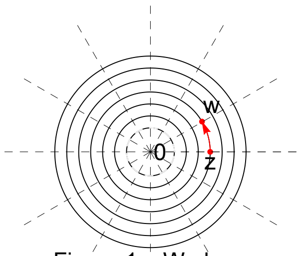

Figure 1a. W plane
Elliptic Möbius transformation
Cf. Fig. [29],[30], pp. 163,165

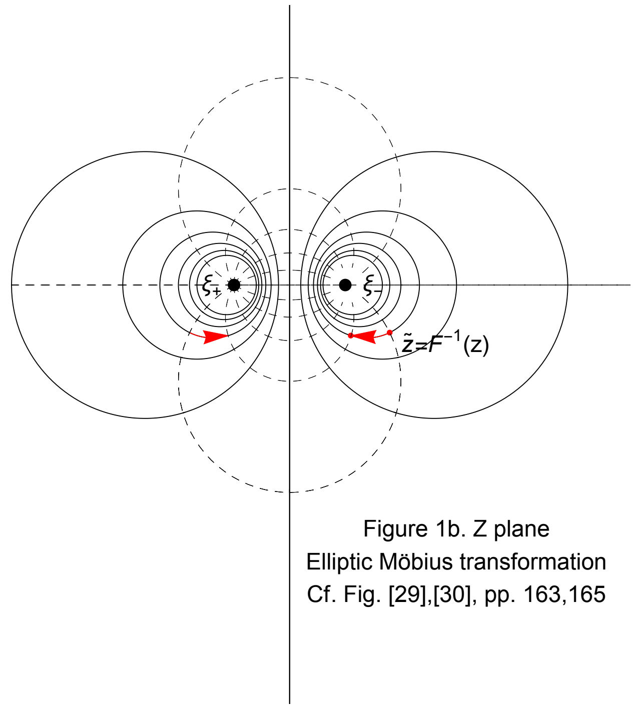

Figure 1b. Z plane
Elliptic Möbius transformation
Cf. Fig. [29],[30], pp. 163,165

Out[992]=

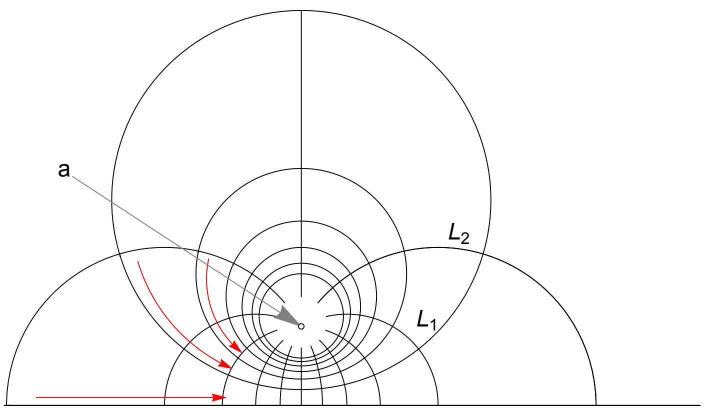

Figure 2. h–rotation
Cf. Figure [27], p. 308

Discussion of h-rotation:

Figure 1b (a variation of Figure [29], p. 163) appears to be a reasonable facsimile of Figure [27], p. 308, illustrating h-rotation. Figure [27] looks like the left half of Figure 1b rotated by -Pi/2. That is, begin with Figure 1a, above, and apply $F^{-1}(z) = (z+1)/(-\xi_+ z - \xi_-)$, where $\xi_+ = -1$ and $\xi_- = 1$ based on an elliptic $\tilde{M} = e^{i\theta} w$ and multiplier $e^{i\pi}$. This results in 1b (after duplicating and reversing the right hand side circles). Then multiply all the points of 1b by -i and retain only the upper half-plane to get Figure 2. The direction of the h-rotation appears to be the same as the Möbius transformation (red arrow) in Figure 1a. Does this mean that the replacement of the fixed points 0 and ∞ in figure 1a with some other points

ξ_+ and ξ_- in 1b can be interpreted as a transform of the complex plane to the hyperbolic plane (rotated by i)? Vasco pointed out (Forum, October 24, 2016) that “Since the fixed points $\xi+$ and $\xi-$ can be any two points, then we can choose $\xi+ = \alpha$i and $\xi- = -\alpha$i, where α is real and positive, and then the picture will be like your figure 2 on page 2.”

To review what the figures represent, [1a] is concentric Euclidean circles and orthogonal rays that represent the possibility of Möbius transformations with fixed points 0 and ∞. F^{-1} takes [1a] with fixed points -1 and 1 to what is labeled an elliptic transformation in [1b] that maps the circles around the fixed points to themselves. The diagram bears a close resemblance to the “elliptic transformation” described in Figure [27] of p. 308, which is classified as “h-rotation”, which is the rotation of hyperbolic lines around a point that seems to correspond to a fixed point of the elliptic transformation. We seem to have two perspectives on the same diagram. Vasco observed “If you look at (46) on page 173, you will see that the Möbius transformation M is the composition of reflections in any two circles which pass through the fixed points [rays---GP]. It then follows from (38) and (37) on page 303 that the h-rotation about a is a perfect rotation (to a Poincarite), as described on pages 308-9.”

The elliptic Möbius transformation of the circles maps the circles centered at point a to themselves. The h-rotations map the orthogonal circles to other orthogonal circles (e.g. L_1 to L_2). But at the same time, the h-rotation must be mapping the circles centered at a to themselves, so the h-rotation is equivalent to an elliptic transformation. The relationship of Figure 2 to Figure 1b is:

h-rotation figure = i (elliptic transformation figure)

with only the upper half-plane retained. It appears that the h-rotation is also equivalent to the Möbius transformations illustrated as rotations with the red arrow in the complex plane of 1a. We obtained the points of the circles on the LHS of [b] by reversing the sign of the real values of the points of the circles on the RHS.

h-translation

In Figure 3, we see that h-translations can also be plotted on Figure 1b, but the translation is along h-circles (only one of which is an h-line) orthogonal to the h-lines. These curves are taken from 1b rotated by Pi/2. We select only the points in the upper half plane. The curves are all equidistant because the angles between them (Pi/6) have equal magnitudes. One reason we know this is because they are Möbius transformations of the rays in Figure 1a.

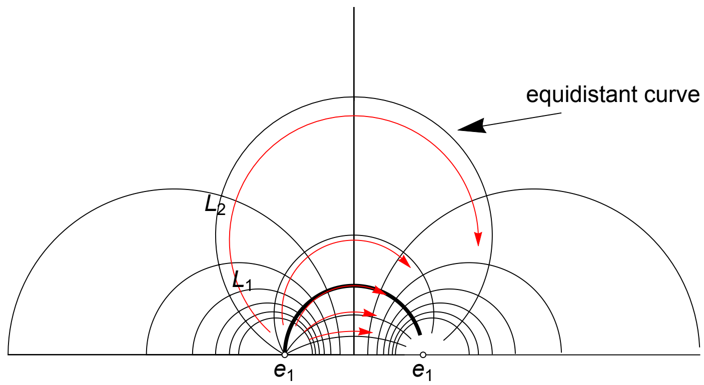

Figure 3. h−translation
Cf. Figure [30], p. 311

Limit Rotation

Out[1019]=

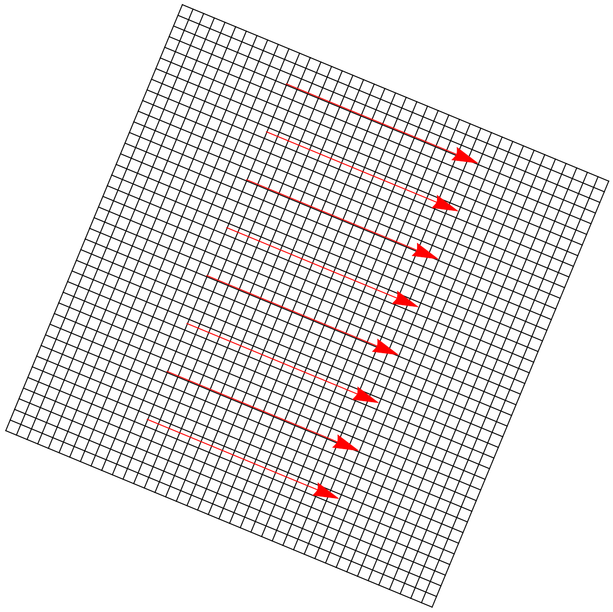

Figure 4a. Complex plane, parabolic transformations
(cf. Figure [33], p. 168)

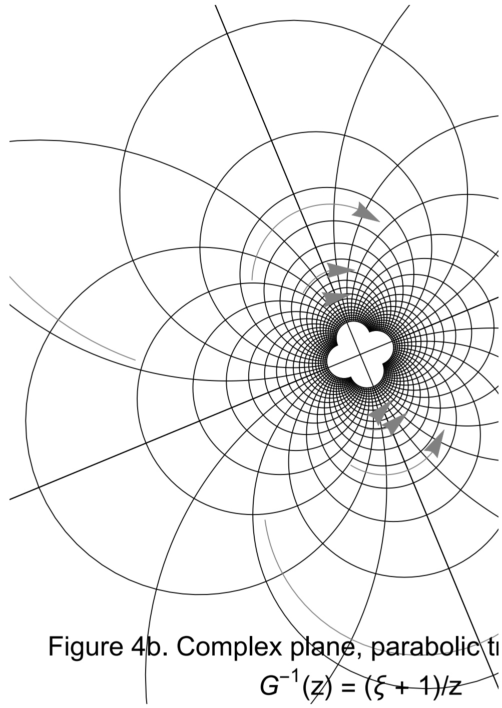

Figure 4b. Complex plane, parabolic t
$G^{-1}(z) = (\xi + 1)/z$

Out[1026]=

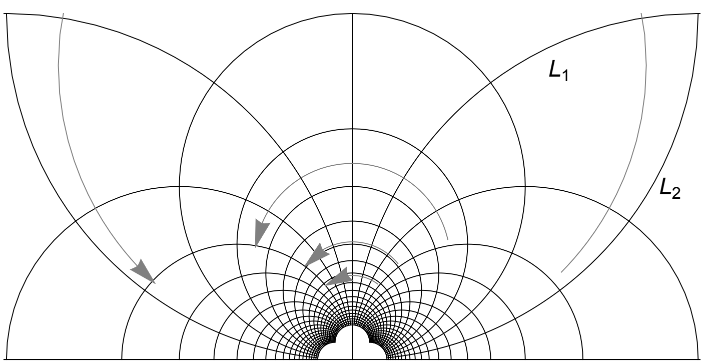

Figure 5. Limit rotation in hyperbolic plane
(cf. Figure [29], p. 310)

Discussion: Figure 5 can be obtained from Figure 4b by a simple rotation and the selection of points from just the upper half-plane (after the appropriate rotation as determined by the direction of the arrows). We don't know why the points in the center region do not go to zero. It will require a different plotting strategy to illustrate the hyperbolic lines and circles in the central region.