Needham, Chapter 5, p. 230
Visual differentiation with a = 1/3.
G. Palmer, December 9, 2015

This discussion pertains to an exercise left to the reader in Chapter 5, Section VI, Visual Differentiation of the Power Function, pp. 229-231. Figure 1 is a response to two sentences in the last paragraph, p. 230:

But if a is a fraction, for example, the both z^a and z^{a-1} are multifunctions possessing many different branches. We urge you to redraw [12] in such a case.

We have let $z = re^{i\theta}$, so that z has three roots. The roots are:

$$\{ r^{1/3}\, e^{i\theta/3},\ r^{1/3}\, e^{i(\theta+2\,\mathrm{Pi})/3},\ r^{1/3}\, e^{i(\theta+4\,\mathrm{Pi})/3}\}$$

For convenience, we let r = 1, but the plot looks much the same with any value of r.

Needham continues:

For example, if a = (1/3) then the infinitesimal arrow on the left will have three images on the right, one for each branch of the cube root function. Unlike the case of the multifunction log(z) ... these images are obtained by amplitwisting the original arrow by three different amounts: each branch of z^a has a different amplitwist. However, your figure will show you [exercise] that

"The amplitwist of each branch of z^a is given by $(z^a)' = a\, z^a / z$, provided that the same branch of z^a is used on both sides of the equation."

Figure 1 can be regarded as an overlay of the two parts of [12], as it plots both z and the three roots. We have drawn a small vector $\vec{\epsilon}$ at z and calculated the amplitwist to be az^a/z. Then since $f(z) = z^{1/3}$, we calculate the amplitwist for each root as

$$f' = \left(z^{1/3}\right)' = (1/3) r^{1/3}\, e^{i(\theta+n2\mathrm{Pi})/3} e^{-i\theta} = \left\{\frac{r^{1/3}\, e^{-i2\theta/3}}{3},\ \frac{r^{1/3}\, e^{i(2\,\mathrm{Pi}-2\,\theta)/3)}}{3},\ \frac{r^{1/3}\, e^{i(4\,\mathrm{Pi}-2\,\theta)/3)}}{3}\right\}$$

where n = 0, 1, 2.

Then the ϵ vectors for the roots are:

$$\vec{\epsilon}_0 = \frac{r^{1/3}\, e^{-i2\theta/3}}{3}\vec{\epsilon}$$

$$\vec{\epsilon}_1 = \frac{r^{1/3}\, e^{i(2\,\mathrm{Pi}-2\,\theta)/3)}}{3}\vec{\epsilon}$$

$$\vec{\epsilon}_2 = \frac{r^{1/3}\, e^{i(4\,\mathrm{Pi}-2\,\theta)/3)}}{3}\vec{\epsilon}$$

Out[4717]=

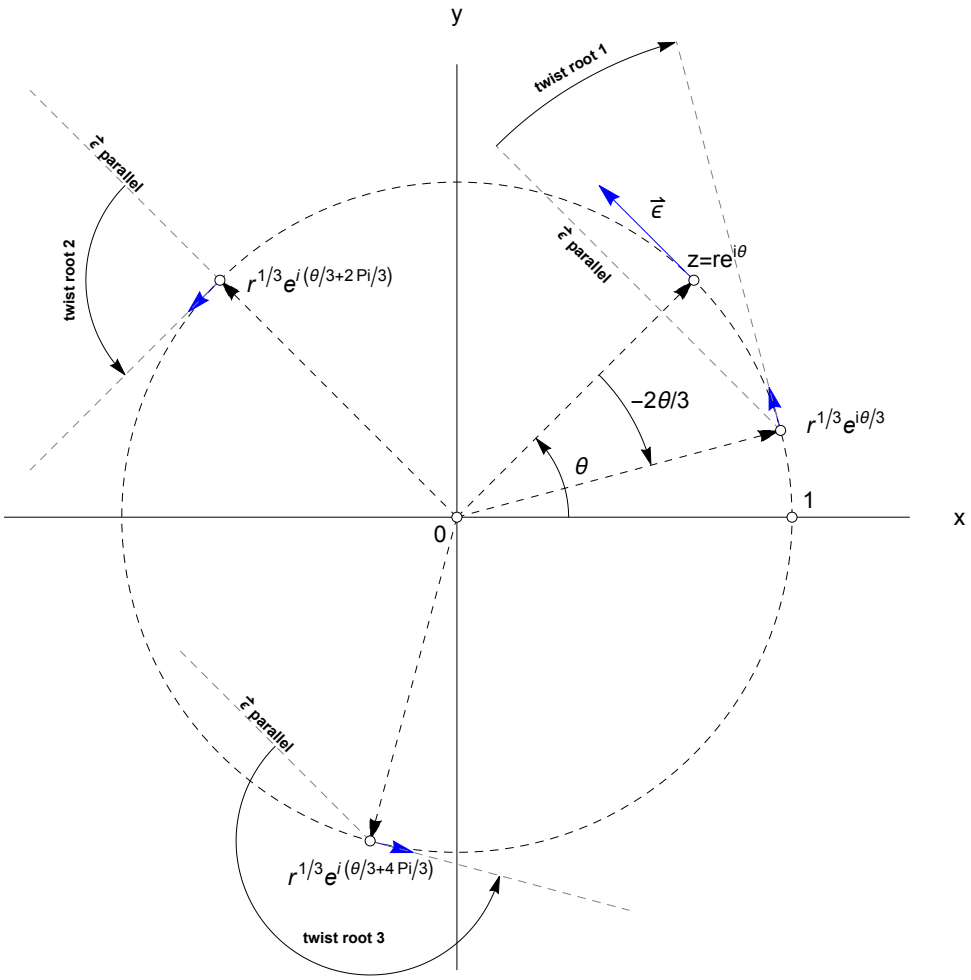


Figure 1. Visual differentiation of $z^{1/3}$