Needham, Chapter 5, XI, 5, Analytic Continuation via Reflections, pp. 252-257
G. Palmer, May 6, 2016 9:30 am

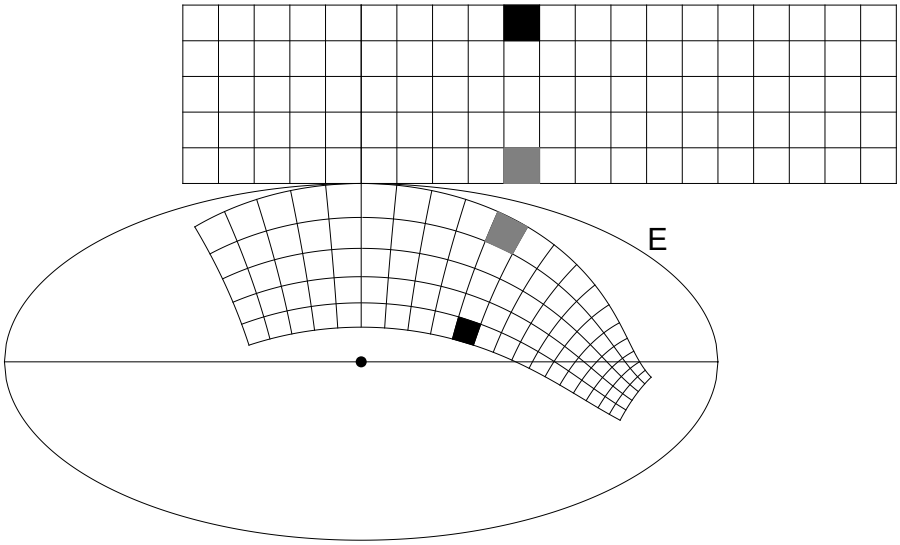


Figure 1. Recreation of Fig. [31], Needham, p. 256

Solve for y and plot the ellipse shown in Figure 1 using the y in (2).

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad (1)$$

$$y = \pm b \left(1 - \left(\frac{x}{a}\right)^2\right)^{1/2} \qquad (2)$$

The exercise from paragraph 2, p. 256.

$S_E(z) = \overline{z}$, so solve the ellipse for $\overline{z}$, where $x = \frac{1}{2}(z + \overline{z})$, $y = \frac{1}{2i}(z - \overline{z})$.

$$\left(\frac{\frac{1}{2}(z+\overline{z})}{a}\right)^2 + \left(\frac{\frac{1}{2i}(z-\overline{z})}{b}\right)^2 = 1 \qquad \text{Substitute into (1)}$$

$$\frac{b^2}{4}(z + \overline{z})^2 - \frac{a^2}{4}(z - \overline{z})^2 = a^2 b^2 \quad \text{Mulitply through by } a^2 b^2 \text{ and by 4}$$

$$b^2(z + \overline{z})^2 - a^2(z - \overline{z})^2 = 4 a^2 b^2$$

Expand the squares

$$(z + \overline{z})^2 = z^2 + 2 z\overline{z} + \overline{z}^2$$

$$(z - \overline{z})^2 = z^2 - 2\, z\overline{z} + \overline{z}^2$$

Return to the equation, substitute the expansions for the squares:

$$b^2\left(z^2 + 2\, z\,\overline{z} + \overline{z}^2\right) - a^2\left(z^2 - 2\, z\,\overline{z} + \overline{z}^2\right) = 4\, a^2\, b^2$$

$$b^2 z^2 + 2\, b^2\, z\overline{z} + b^2\overline{z}^2 - a^2 z^2 + 2\, a^2\, z\,\overline{z} - a^2\overline{z}^2 = 4\, a^2\, b^2 \qquad \text{expand}$$

$$\left(b^2 z^2 - a^2 z^2\right) + \left(2\, b^2\, z\overline{z} + 2\, a^2\, z\,\overline{z}\right) + \left(b^2\overline{z}^2 - a^2\overline{z}^2\right) = 4\, a^2\, b^2 \qquad \text{group like terms}$$

$$\left(b^2 - a^2\right)z^2 + 2\left(a^2 + b^2\right) z\overline{z} + \left(b^2 - a^2\right)\overline{z}^2 = 4\, a^2\, b^2 \qquad \text{factor}$$

$$\left(b^2 - a^2\right)\overline{z}^2 + 2\left(a^2 + b^2\right) z\overline{z} + \left(b^2 - a^2\right)z^2 - 4\, a^2\, b^2 = 0 \qquad \text{subtract } 4\, a^2\, b^2 \text{ frm both sides}$$

$$\overline{z}^2 + 2\,\frac{a^2+b^2}{b^2-a^2}\, z\overline{z} + z^2 - \frac{4\, a^2\, b^2}{b^2-a^2} = 0 \qquad \text{divide through by } \left(b^2 - a^2\right)$$

$$\overline{z} = \frac{-2\,\frac{a^2+b^2}{b^2-a^2}\, z \pm \sqrt{4\left(\frac{a^2+b^2}{b^2-a^2}\, z\right)^2 - 4\left(z^2 - \frac{4\, a^2\, b^2}{b^2-a^2}\right)}}{2}$$

quadratic equation.

$$\overline{z} = \frac{a^2+b^2}{a^2-b^2}\, z \pm \sqrt{\left(\left(\frac{a^2+b^2}{-\left(a^2-b^2\right)}\, z\right)^2 - \left(\frac{-\left(a^2-b^2\right)}{-\left(a^2-b^2\right)}\right)^2 z^2 - 4\,\frac{\left(a^2-b^2\right) a^2\, b^2}{\left(-\left(a^2-b^2\right)\right)^2}\right)}$$

create common denom. $\left(a^2 - b^2\right)^2$

$$= \frac{1}{a^2-b^2}\left[\left(a^2 + b^2\right)z \pm \sqrt{\left(\left(a^2 + b^2\right)^2 z^2 - \left(a^2 - b^2\right)^2 z^2 - 4\left(a^2 - b^2\right) a^2\, b^2\right)}\right]$$

move $\frac{1}{\left(a^2-b^2\right)^2}$ outside [ ]

$$= \frac{1}{a^2-b^2}\left[\left(a^2 + b^2\right)z \pm \sqrt{\left(\left(a^4 + 2\, a^2\, b^2 + b^4\right) z^2 - \left(a^4 - 2\, a^2\, b^2 + b^4\right) z^2 - 4\left(a^2 - b^2\right) a^2\, b^2\right)}\right]$$

by expansion

$$= \frac{1}{a^2-b^2}\left[\left(a^2 + b^2\right)z \pm \sqrt{\left(a^4\, z^2 + 2\, a^2\, b^2\, z^2 + b^4\, z^2 - a^4\, z^2 + 2\, a^2\, b^2\, z^2 - b^4\, z^2 - 4\left(a^4\, b^2 - a^2\, b^4\right)\right)}\right]$$

by expansion

$$= \frac{1}{a^2-b^2}\left[\left(a^2 + b^2\right)z \pm \sqrt{4\, a^2\, b^2\, z^2 - 4\left(a^4\, b^2 - a^2\, b^4\right)}\ \right]$$

by adding like terms

$$= \frac{1}{a^2-b^2}\left[\left(a^2 + b^2\right)z \pm \sqrt{4\, a^2\, b^2\, z^2 + 4\left(a^2\, b^4 - a^4\, b^2\ \right)}\right.$$

$$= \frac{1}{a^2-b^2}[(a^2+b^2)z \pm 2ab\sqrt{z^2+b^2-a^2}$$

The points to the left and right of the y axis require different Schwarzian reflections. A negative square root must be taken for the left side. Also, in our programming script, the Schwarzian reflection produces an anomaly on the x = 0 point of every horizontal line to the right of the y axis, sending the point to ~ (0 - 4.3 i), which shows up on the plot as a vertical line from -4i to 0. Those six points were dropped from the plot. They were not needed because the left hand side also produced a vertical line at {0, y}.

Now derive $\mathcal{R}$ from $\mathcal{S}$.

$\mathcal{R}_K(z) = \overline{\mathcal{S}_K(z)}$, from p. 255, paragraph 2

$\mathcal{R}_E(z) = \overline{\mathcal{S}_E(z)} = \frac{1}{a^2-b^2}[(a^2+b^2)z \pm 2ab\sqrt{z^2+b^2-a^2}$, where a = 2, b = 1

$$= \frac{1}{2^2-1^2}[(2^2+1^2)\overline{z} \pm 4\overline{\sqrt{z^2+1^2-2^2}}$$

$$= \frac{1}{3}[5\overline{z} \pm 4\overline{\sqrt{z^2-3}}\,] = \frac{1}{3}[5\overline{z} \pm 4\sqrt{\overline{z^2-3}}\,] \qquad \text{because } \overline{\sqrt{z}} = \sqrt{\overline{z}} \ ^*$$

$$= \frac{1}{3}[5\overline{z} \pm 4\sqrt{\overline{z}^2-3}\,]$$

Strangely, Needham omitted the positive branch of both the general and the specific $\mathcal{R}_E(z)$.

The plot just applies $\mathcal{R}_E(z)$ to the squares outside the ellipse. The equation used for $\mathcal{R}_E(z)$ is the last equation presented in XI, 5, p. 256. In *Mathematica* it takes the form

```
sReflLeft[z_]:=(1/3)(5 Conjugate[z] + 4 Sqrt[ Conjugate[z]^2 - 3]);      z left of y axis

sReflRight[z_]:=(1/3)(5 Conjugate[z] - 4 Sqrt[Conjugate[z]^2 - 3]);     z right of y axis
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For $\overline{z}$ read Conjugate[z].

When the side of a square has size .2, the plot resembles that of Figure [31], but changing the size of the squares produces different effects.

* $\overline{\sqrt{z}}\,\sqrt{z} = \left|\sqrt{z}\right|^2 = |z| = |\overline{z}| = |\sqrt{\overline{z}}\,|^2 = \sqrt{\overline{z}}\,\overline{\sqrt{\overline{z}}}$

$\overline{\sqrt{z}} = \overline{\sqrt{re^{i\theta}}} = \overline{\pm\sqrt{r}\,e^{i\theta/2}} = \pm\sqrt{r}\,e^{-i\theta/2} = \sqrt{re^{-i\theta}} = \sqrt{\overline{z}}$