Needham, ch 5, pp. 256-257.
G. Palmer, May 29, 2016

Out[2663]=

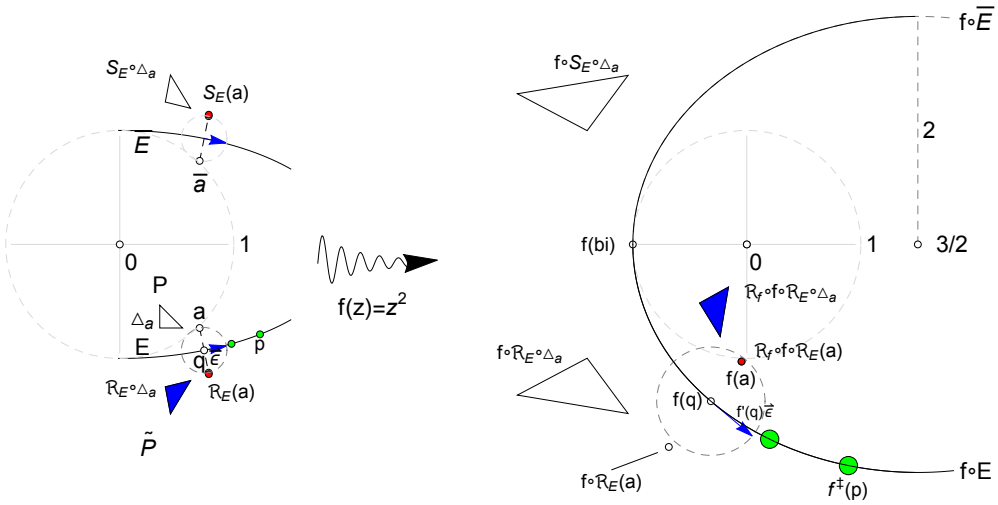


Figure 3. Continuation $f^{\ddagger}$ of f from P to $\tilde{P} = \mathcal{R}_E(P)$:
$f(z) = z^2$, $f^{\ddagger} = \mathcal{R}_f \circ f \circ R_E$.

The discussion and Figure 3 pertain to the last two paragraphs of Chapter 5, Section XI, p. 256-257. Figure 3 is very much like Figure [30], p. 255. Additionally, it applies the analytic function z^2 to the segment E of an ellipse to obtain a second segment which we label f∘E. In this case E is a segment of the ellipse where $x \geq 0$ and $y \leq 0$. Beginning with point a on the lower left, a is reflected in E to point $\mathcal{R}_E(a)$, the red point which falls on the ordinary reflection $\tilde{a}$ (not shown) around the tangent. Then f∘$\mathcal{R}_E(a)$ sends it to the RHS and $f^{\ddagger} = \mathcal{R}_f \circ f \circ \mathcal{R}_E(a)$ reflects f∘$\mathcal{R}_E(a)$ in the segment f∘E. So as not to overtax *Mathematica*'s text graphics, we write $\mathcal{R}_f$ rather than $\mathcal{R}_{\hat{E}}$. We require also that $f^{\ddagger}$ = f on L. If all this works as it should, we have a continuation $f^{\ddagger}$ of f to $\tilde{P} = \mathcal{R}_E(P)$, where P in this case is the region inside the ellipse to which the segment E belongs. P is the region containing a.

Derivation of Equation for $\hat{E}$ (Ellipse on RHS). This derivation follows a document by Vasco, but with variations in notation.

Let $z = x + iy$. From p. 88, we can write the new ellipse as $w = u + iv = z^2$. Then

$$x = a \cos t, \quad y = b \sin t$$

$$w = x^2 - y^2 + 2ixy$$

$$u = x^2 - y^2 = a^2 \cos^2 t - b^2 \sin^2 t \qquad (1)$$

$$v = 2xy = 2ab \cos t \sin t \qquad (2)$$

$$w = a^2 \cos^2 t - b^2 \sin^2 t + i(2ab \cos t \sin t) \qquad (3)$$

From trig identities,

$$\cos^2 t = \frac{\cos 2t + 1}{2}$$

$$\sin^2 t = \frac{\cos 2t - 1}{2}$$

$$2 \cos t \sin t = \sin 2t$$

Using these substitutions rewrite (1) and (2) and rearrange them to solve for cos 2t and sin 2t.

$$u = a^2\left(\frac{\cos 2t + 1}{2}\right) - b^2\left(\frac{1 - \cos 2t}{2}\right)$$

$$2u = a^2 \cos 2t + a^2 - b^2 + b^2 \cos 2t$$

$$2u = \left(a^2 + b^2\right) \cos 2t + a^2 - b^2$$

$$\left(a^2 + b^2\right) \cos 2t = 2u - \left(a^2 - b^2\right)$$

$$\cos 2t = \frac{2u - \left(a^2 - b^2\right)}{\left(a^2 + b^2\right)}$$

$$v = 2ab \cos t \sin t$$

$$= ab \sin 2t$$

$$\sin 2t = \frac{v}{ab}$$

Using $\cos^2 + \sin^2 = 1$,

$$\left[\frac{u - \left(a^2 - b^2\right)/2}{\left(a^2 + b^2\right)/2}\right]^2 + \left[\frac{v}{ab}\right]^2 = 1 \qquad (4)$$

Given that a = 2, b = 1, this is the equation for a horizontal ellipse with center at $\left(a^2 - b^2\right)/2 = 3/2$, a horizontal semi-major axis of $\frac{\left(a^2 + b^2\right)}{2} = 5/2$, and a vertical semi-minor axis of ab = 2.

We solve for v for the purpose of plotting.

$$\left[\frac{v}{ab}\right]^2 = 1 - \left[\frac{u - \left(a^2 - b^2\right)/2}{\left(a^2 + b^2\right)/2}\right]^2$$

$$\frac{v}{ab} = \left(1 - \left[\frac{u - (a^2 - b^2)/2}{(a^2 + b^2)/2}\right]^2\right)^{1/2}$$

$$v = a\,b\,\left(1 - \left[\frac{u - (a^2 - b^2)/2}{(a^2 + b^2)/2}\right]^2\right)^{1/2}$$

To plot the elliptical segment with function v(u), vary u from Pi/2 to Pi on the positive root. This plots the same curve segment that we obtained from plotting z^2 on the points of E. The segment can also be plotted with equation (3) above. All three equations produce congruent elliptical segments.

Find the Schwarzian function by substituting into (4) with two complex identities, as explained on p. 256.

$$u = \frac{w + \overline{w}}{2}$$
$$v = \frac{w - \overline{w}}{2i}$$

$$\left[\frac{\frac{w + \overline{w}}{2} - (a^2 - b^2)/2}{(a^2 + b^2)/2}\right]^2 + \left[\frac{w - \overline{w}}{2\,\text{iab}}\right]^2 = 1 \qquad \text{(by substitution for u and v into (4))}$$

$$\left[\frac{w + \overline{w} - (a^2 - b^2)}{(a^2 + b^2)}\right]^2 + \left[\frac{w - \overline{w}}{2\,\text{iab}}\right]^2 = 1$$

To obtain $S_f(w)$, solve for $\overline{w}$, with a = 2 and b = 1. Since it is late, I solve using *Mathematica*. What would Euler and Gauss think? But's it's only for plotting, not for proving.

$$S_f(w) = \overline{w} = \frac{1}{9}\left(-48 + 41\,w + 40\sqrt{-3\,w + w^2}\right)$$

Then $\Re_f(w) = \overline{1/9\left(-48 + 41\,w + 40\sqrt{-3\,w + w^2}\right)}$. A function based on this equation is used in plotting the reflections on the RHS in Figure 3. The red dot on the lower right is $f^{\ddagger} = \Re_f \circ f \circ \Re_E(a)$. Compare the equation for $f^{\ddagger}$ in Needham, p. 257.

Conformality/Analycity

Both a and the triangle $\triangle_a$ appear to be reflected in the curve E, and the small blue reflected triangle, $\Re_E \circ \triangle_a$, appears to be similar to the initial triangle, $\triangle_a$. The two triangles have opposite orientations to the infinitesimal tangent ϵ, showing that $\Re_E$ is anti-conformal. The application of f to $\Re_E \circ \triangle_a$ and $S_E \circ \triangle_a$ produces images similar to the sources with the same orientation, showing that f is analytic and conformal. The two triangles $\triangle_a$ and $\Re_f \circ f \circ \Re_E \circ \triangle_a$ appear to be conformal. This is a good result because f is conformal and the two reflections together are conformal. The green points that fall on the curve E also fall as they should on the curve f∘E when reflected with $f^{\ddagger} = \Re_f \circ f \circ \Re_E(z)$. The size of the green points is not significant.