Needham, Chapter 5, Sections IX.3, X.5, Miscelaneous exercises in text, p. 239-246
G. Palmer, April 15, 2016

Derivation of proportionalities § IX.3, p. 239.

Show that $\chi \propto f'\mathcal{K} \propto \overline{if'} \propto if'$.

Let α and β be constants. $\tilde{\kappa}$ is a constant. To show $x \propto y$ is to show that $x = \alpha y$ or $x = \beta y$. $\chi \propto f'\mathcal{K}$ because ξ is in the direction of $\mathcal{K}$ and $\chi = f''\xi$ from p. 237. Then

$$f'\mathcal{K} = f'' \, i\left(\overline{\frac{f''}{f'}}\right)\frac{1}{|f'|} = \frac{i|f''|^2}{\bar{f}'}\frac{1}{|f'|} = \frac{i|f''|^2}{|f'|}\frac{1}{\bar{f}'} = \alpha\frac{i}{\bar{f}'}$$

$$= \alpha\frac{if'}{\bar{f}'f'} = \alpha\frac{if'}{|f'|^2} = \beta\, if'$$

QED

[exercise], § IX.3, p. 241. Show that $\mathcal{E} = \hat{\xi} \times \mathcal{K}$.

$\mathcal{E}$ is the rate of increase of the linear size of $\tilde{Q}$ as it moves along in an arbitrary direction $\hat{\xi}$. We can write $\hat{\xi}$ as $e^{i\theta}$. The last paragraph on p. 239 suggests that an analytic mapping f is responsible for the translation of $\tilde{Q}$ along $\hat{\xi}$. Then the rate of increase of $\tilde{Q}$ must be the amplitude |f'|. We know that $\mathcal{K} = i\left(\overline{\frac{f''}{f'}}\right)\frac{1}{|f'|}$, so

$$|f'|\,\mathcal{K} = i\left(\overline{\frac{f''}{f'}}\right)$$

$$|f'| = i\left(\overline{\frac{f''}{f'}}\right)\frac{1}{\mathcal{K}}$$

$$\hat{\xi} \times \mathcal{K} = |\hat{\xi}|\,|\mathcal{K}| \sin\theta = |\mathcal{K}|\,\mathrm{Im}[\hat{\xi}]$$

We can see that $|\mathcal{K}|$ is at it's greatest when $\hat{\xi} = i$ (because Im[i] = 1, it's highest possible value]. This must mean that the direction of $\mathcal{K} = i$. But when $\hat{\xi} = e^{i\theta} = i$, θ = pi/2 and sin θ = 0, so $\hat{\xi} \times \mathcal{K}$ is at its lowest value. $\hat{\xi} \times \mathcal{K}$ must then reach its highest value when $\hat{\xi} = 1$, the direction of $-i\mathcal{K}$.

We should show that

$$|f'| = |\mathcal{K}|\,\mathrm{Im}[\hat{\xi}]$$

$$\mathcal{K} = i\left(\overline{\frac{f''}{f'}}\right)\frac{1}{|f'|},$$

so

$$|\mathcal{K}| = \frac{i}{|f'|}\left|\overline{\left(\frac{f''}{f'}\right)}\right|$$

and

$$|f'| = |\hat{\xi}|^2\,|\mathcal{K}|\sin\theta$$

$$= |\hat{\xi}|^2\,\frac{i}{|f'|}\left|\overline{\left(\frac{f''}{f'}\right)}\right|\sin\theta$$

$$= \frac{i}{|f'|}\left|\overline{\left(\frac{f''}{f'}\right)}\right|\sin\theta$$

$$= \alpha\sin\theta$$

[exercise], § X.5, p. 246. [Corrected from March 16]

To find $\tilde{\kappa}$, simply put f(z) = z^2 into (23) and thereby obtain [exercise]

$$\tilde{\kappa} = \frac{1}{2}\left[\frac{\cos\gamma}{r^2} + \frac{k}{r}\right]$$

The solution depends on finding an expression for γ.

Refer to Figure [24], p. 245. Since f = z^2, f' = 2z = 2p = 2 $re^{i\theta}$ and |f'(p)| = 2r.

twist = Arg(f'(p))

Then

arg(v) = arg(f(p)) + twist = θ + θ = 2θ.

γ = arg(v_t) − arg(v) = (θ + Pi/2) - 2θ

= Pi/2 - θ

θ = Pi/2 - γ

sin θ = sin(Pi/2-γ) = cos γ (1)

$$\tilde{\kappa} = \frac{1}{|f'(p)|}\left(\mathrm{Im}\left[\frac{f''(p)\,\hat{\xi}}{f'(p)}\right] + \kappa\right) \qquad (23)$$

$\hat{\xi}$ is a unit vector tangent to the curve f at p, so it has the same angle as v. We rewrite it as $e^{i2\theta}$.

$$\tilde{\kappa} = \frac{1}{2r}\left(\text{Im}\left[\frac{2\,e^{i2\theta}}{2\,\text{re}^{i\theta}}\right] + \kappa\right)$$

$$= \frac{1}{2r}\left(\text{Im}\left[\frac{e^{i\theta}}{r}\right] + \kappa\right)$$

$$= \frac{1}{2r}\left(\frac{\sin\theta}{r} + \kappa\right), \text{ by Euler's equation}$$

$$= \frac{1}{2}\left(\frac{\cos\gamma}{r^2} + \frac{\kappa}{r}\right), \text{ from (1) and by rearranging}$$

Done.

§ X.5(35), p. 246.
Needham did not assign this one as an exercise for the reader, but it seemed worthwhile going through it.

Given

$$\text{v} = h\left(\frac{\sec\gamma}{r}\right) \qquad \text{(1) [Needham (33)]}$$

$$\text{F} = h^2\left(\frac{\kappa\sec^3\gamma}{r^2}\right) \qquad \text{(2) [Needham (34)]}$$

$$\tilde{F} = \tilde{h}^2\left(\frac{\tilde{\kappa}\sec^3\gamma}{\tilde{r}^2}\right) \qquad \text{(3) [Needham, p. 246]}$$

$$\tilde{\kappa} = \frac{1}{2}\left[\frac{\cos\gamma}{r^2} + \frac{\kappa}{r}\right] \qquad \text{(4) [Needham, p. 246]}$$

Derive

$$\tilde{F} = \left(\frac{\tilde{h}}{h}\right)^2\frac{\left[\frac{1}{2}v^2 + \frac{1}{2}\text{rF}\right]}{\tilde{r}^2} \qquad \text{[Needham (35)]}$$

Begin

$$\tilde{F} = \tilde{h}^2\left(\frac{\frac{1}{2}\left[\frac{\cos\gamma}{r^2} + \frac{\kappa}{r}\right]\sec^3\gamma}{\tilde{r}^2}\right) \quad \text{(4) into (3)}$$

$$= \tilde{h}^2\left(\frac{\frac{1}{2}\left[\frac{\cos\gamma\sec^3\gamma}{r^2} + \frac{\kappa\sec^3\gamma}{r}\right]}{\tilde{r}^2}\right) \qquad \text{move } \sec^3\gamma \text{ inside brackets}$$

$$\frac{\kappa\sec^3\gamma}{r} = \frac{\text{rF}}{h^2} \qquad \text{rearrangment of (2)}$$

$$= \tilde{h}^2 \left(\frac{\frac{1}{2}\left[\frac{\cos\gamma \sec^3\gamma}{r^2} + \frac{rF}{h^2} \right]}{\tilde{r}^2} \right) \quad \text{sub. frm previous}$$

$\sec \gamma = rv/h$ rearrangement of (1)

$\sec^2 \gamma = r^2 v^2 / h^2$

$\cos \gamma \sec \gamma = 1$

$$\tilde{F} = \tilde{h}^2 \left(\frac{\frac{1}{2}\left[\frac{\cos\gamma \sec\gamma\, r^2 v^2}{r^2 h^2} + \frac{rF}{h^2} \right]}{\tilde{r}^2} \right) \text{ sub. frm previous}$$

$$= \left(\frac{\tilde{h}}{h} \right)^2 \left(\frac{\frac{1}{2} v^2 + \frac{1}{2} rF}{\tilde{r}^2} \right) \quad \text{by sub. and rearrangement}$$

This equation for $\tilde{F}$ does not require $\tilde{\kappa}$ or γ, but it does require v, r, and F.

Figure 1 is just an attempt to illustrate some of the values discussed in Section X.5.
The figure uses f(z) = z^2 to illustrate complex curvature and the values from which it is derived. $\mathcal{K}$ is plotted at two locations, at f(p) and at the origin. $\tilde{\kappa}$ (orange) is plotted as a vector orthogonal to $\tilde{\xi}$, and drawn from the end of $\tilde{\xi}$. I'm not sure whether this is an useful way to display it.

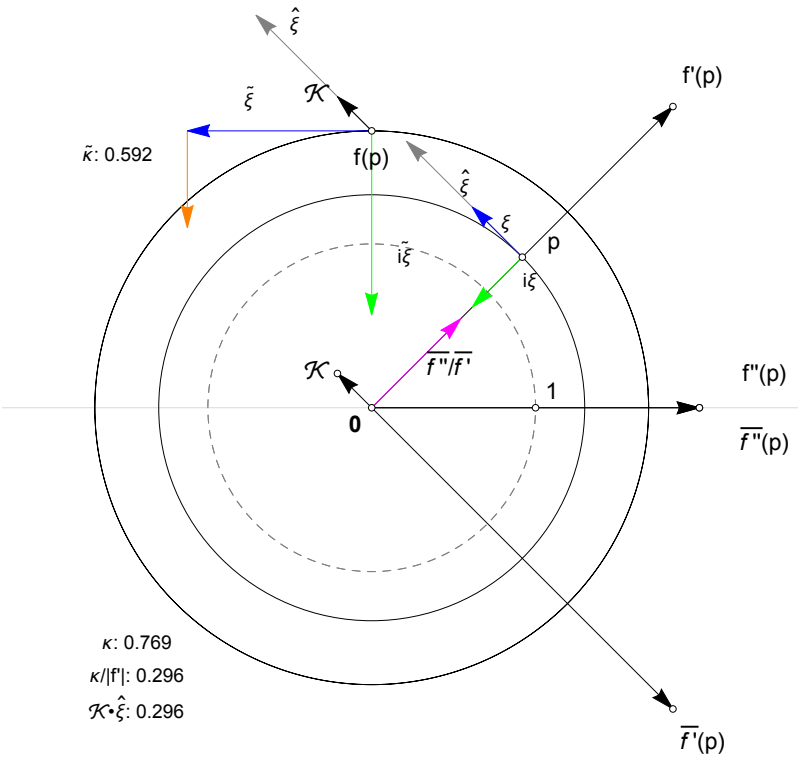


Figure 1. Curvature: $\tilde{\kappa} = \mathcal{K}\cdot\hat{\xi} + \kappa/|f'(p)|$
$\mathcal{K} = i\overline{f''}/(\overline{f'}|f'|)$, $f(z) = z^2$