Needham, Chapter 5, Exercise 9, p. 258.
G. Palmer, January 4, 2015

9. Derive the quotient rule in an analogous way to the product rule (see page 226). [Hint: Multiply top and bottom of (A/B) by (b-ξg').]

A = a+ξf' and B = b+ξg'

A/B = (a+ξf')/(b+ξg') = (a+ξf')/(b+ξg') (b-ξg')/(b-ξg')

= [(a+ξf')(b-ξg')]/[(b+ξg')(b-ξg')]

$= (ab - a\xi g' + b\xi f' - \xi^2 f' g') \big/ (b^2 - \xi^2 g'^2)$

$\simeq$ $(ab - \xi(ag' - bf'))/b^2$, by ignoring the infinitesimal products

$= (ab + \xi(bf' - ag')/b^2$ (where $\xi(bf' - ag')/b^2$ is the image of ξ)

From this we deduce the quotient rule:

$(f/g)' = (bf' - ag')/b^2$

By construction, f = a and g = b (Figure 1). Then

$(f/g)' = (gf' - fg') \big/ g^2$

which is the quotient rule (Stewart J. *Calculus*, 2nd Ed., Brooks/Cole, Pacific Grove, p. 120).

Figure 1 plots some of the points and vectors involved using f = log|z| + iArg(z) and g = z^2. It also shows that f, g, and f/g are conformal.

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(* SCRIPT FOR FIGURE 1 *)

Clear["Global`*"];

(* CONSTANTS *)

pr = 2;

z = 1 + I;

ϵ = .2 z;


(* FUNCTIONS *)

f[z_] := Log[Abs @ z] + I Arg @ z;

g[z_] := z^2;
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(* derivatives of f and g *)

df[z_] := 1 / z;

dg[z_] := 2 z;

(* the solution to the problem *)

dQuofg[z_] := (df[z] g[z] - f[z] dg[z] ) / g[z]^2 ;

(* vector to complex *)

v[z_] := {Re @ z, Im @ z};

SetAttributes[{v}, Listable];


(* POINTS *)

ptSz = .01;

pts = {EdgeForm@Black, FaceForm@White,
    Table[Disk[v @ pt, ptSz], {pt,
      {0, z, f @ z, g @ z, f @ z / g @ z}
     }]};


(* ARROWS *)

arrowData = {
    {0, z},
    {0, f @ z},
    {0, g @ z},
    {0, (f @ z ) / (g @ z)}
  };

ϵArrowData = {
    {z, z + ϵ},
    {f @ z, f @ z + ϵ / z},
    {g @ z, g @ z + 2 z ϵ },
    {(f @ z ) / (g @ z), (f @ z ) / (g @ z) + dQuofg[z] ϵ}
  };

ϵiArrowData = {
    {z, z + I ϵ},
    {f @ z + (ϵ / z) I ϵ} ,
    {f @ z, f @ z + I ϵ / z}, {g @ z, g @ z + 2 z I ϵ},
    {(f @ z ) / (g @ z), (f @ z ) / (g @ z) + dQuofg[z] I ϵ}
  };

zfgArrows = {Arrowheads @ .03, Table[Arrow[v @ pts], {pts, arrowData}]};

ϵArrows = {Blue, Arrowheads @ .03, Table[Arrow[v @ pts], {pts, ϵArrowData}]};

ϵiArrows =  {Green, Arrowheads @ .03, Table[Arrow[v @ pts], {pts, ϵiArrowData}]};
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```
arrows = {zfgArrows, ϵArrows, ϵiArrows};


(* LINES *)

hzLine = {Dotted, Gray, Line[v @ {I, 1 + I}]};

vzLine = {Dotted, Gray, Line[v @ {1, 1 + I}]};

lines = {hzLine, vzLine};


(* TEXT *)

small = 8;
default = 12;
large = 16;

data = {
    {"0", -.1 - .1 I, default},
    {"ϵ", z + .15 z + .1, small},
    {"iϵ", z + .15 I z + .1, small},
    {"z", z + .07, default},
    {"f", f @ z - .05, default},
    {"g", g @ z + .1, default},
    {"f'(z)ϵ", f @ z + ϵ / z - .05 + .05 I, small},
    {"g'(z)ϵ", g @ z + 2 z ϵ / 2 + .2, small},
    {"f/g", (f @ z) / (g @ z) + .1 - .03 I, default},
    {"f(z) = ln|z| + i Arg(z)\ng(z) = z^2", 1 + 2.5 I, default},
    {"((gf' - fg') / g^2)ϵ = (f/g)' ϵ", -.2 - .2 I, small},
    {"Figure 1. Quotient rule", 1 - .5 I, large}
  };

text = Table[Text[Style[item[[1]], item[[3]]], v @ item[[2]]], {item, data}];

(* GRAPHICS *)

Graphics[{pts, arrows, lines, text}, Background → White,
 PlotRange → {{-pr / 2, pr}, {-.5 pr, 1.5 pr}}, Axes → True]
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