Needham, Chapter 5, Exercise 6, p. 258.
G. Palmer, January 2, 2015

7. Use the Cartesian CR equations to show that the only mapping that sends parallel lines to parallel lines is the linear mapping. [Hint: Begin with the case of horizontal lines being mapped to horizontal lines. How does this translate into 'Equationspeak'? Now solve CR.]

The Cartesian CR equations are:

$$\partial_x u = \partial_y v \qquad \text{and} \qquad \partial_x v = -\partial_y u$$

Since we are beginning with horizontal lines, we have

$$\partial_x v = -\partial_y u = 0$$

We fall back on the first equation, recognizing that the LHS depends only on x and the RHS depends only on y.

$$\partial_x U(x) = \partial_y V(y)$$

This can only be generally true if both sides equal a (complex) constant, say A (after p. 221). Needham would say, "dispensing with the superfluous partial derivatives, we thereby obtain"

$$\frac{dU}{dx} = A, \qquad \frac{dV}{dy} = A$$

Integrate both sides:

$$U = Ax + \text{const.}, \qquad V = Ay + \text{const};$$

$$f = Ax + iAy + B, \text{ where B is a complex constant}$$

$$= A(x+iy) + B$$

$$= Az + B$$

In the case of horizontal lines, z consists only of x.

Now consider that rotating a set of horizontal lines by a fixed angle will result in a set of parallel lines at whatever angle is desired.

$$g = e^{i\theta} f = e^{i\theta}(Ax + B)$$

$$= e^{i\theta} Ax + e^{i\theta} B$$

which has the general form

$$g = Cz + D$$

where C and D are complex constants. The form $e^{-i\theta}$g rotates the parallel lines at θ back to horizontal lines. The general form of g is equivalent to that of f = Az + B. Therefore, the linear mapping sends parallel lines to parallel lines.

Prove that *the only mapping that sends parallel lines to parallel lines is the linear mapping*. We try a proof by contradiction. Suppose there is a nonlinear mapping h = U(x,y) + iV(x,y) that sends parallel lines to parallel lines. Let Z be a set of parallel lines at any angle with the real line and apply h(Z) to obtain a new set of parallel lines. Translate the set h(Z) so that a line L in h(Z) intersects the origin. Then if L makes angle ϕ with the x axis, $e^{-i\phi}$h(z) sends all the lines to horizontal lines.

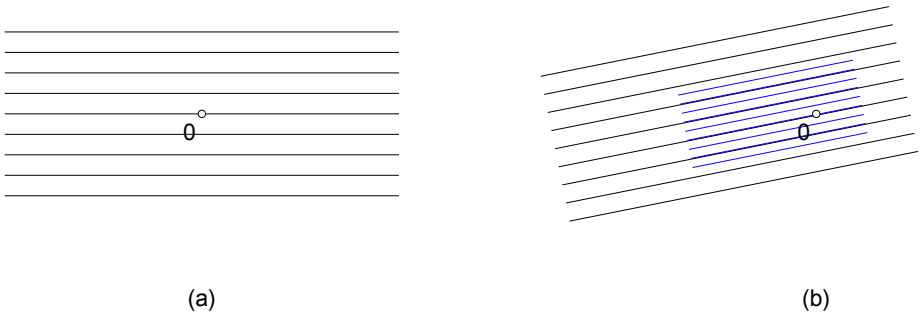
Since $e^{-i\phi}$h(z) is nonlinear, we write its partials wrt x and y as follows:

$$e^{-i\phi}\partial_x U(x, y),\ \ e^{-i\phi}\partial_x V(x,y)$$

$$e^{-i\phi}\partial_y U(x, y),\ \ e^{-i\phi}\partial_y V(x, y)$$

We know that $\partial_x v = -\partial_y u = 0$, because the lines are horizontal and $e^{-i\phi}$ is non-zero. Because the lines are horizontal, we know that V is a constant. Then, to satisfy the CR equation $\partial_x U(x, y) = \partial_y V(x, y)$, U must also equal a constant, which after integration, leads us to the linear mappings $e^{-i\phi}$h = $e^{-i\phi}$(Az + B) and h = (Az + B), which is equivalent to the linear mappings f and g. Hence, the existence of a non-linear mapping that sends parallel lines to parallel lines is a contradiction.

Figure 1 shows the mapping of (a) to (b) under g = $e^{i3Pi/4}$(Az + B) where A = 1/2+3i/4 and B = 1/2+i/2. The group of blue lines plots (r g), where r is a real number.



(a) (b)

Figure 1. g = $e^{i\theta}$(Az+B)