Needham, Chapter 5, Exercise 6, p. 258.
G. Palmer, December 29, 2015

6. Solve the Polar CR equations given that $\partial_\theta v \equiv 0$. Express your answer in terms of a familiar function, and interpret everything you have done geometrically.

The question is ambiguous between the Polar-Cartesian equations and the Polar-Polar equations. As $\partial_\theta v$ has polar variables in the preimage plane and cartesian variables in the image plane, the Polar-Cartesian form seems most appropriate.

Since, $\partial_\theta v \equiv 0$, $r\partial_r u$ must also equal zero by (4), p. 218. Varying θ produces no change in v, so rotating the point only moves the image horizontally. Varying r produces no change in u, so moving the point radially outward only moves the image vertically.

The second Polar-Cartesian equation is $\partial_\theta u = -\partial_r v$. From the previous paragraph, we know that u depends only on θ and v depends only on r. We rewrite the equation:

$$\partial_\theta U(\theta) = -r\,\partial_r V(\mathrm{r})$$

Both sides are real quantities, so both sides must equal a constant, say A, as on p. 20.

$$\partial_\theta U(\theta) = \mathrm{A}, \qquad -r\,\partial_r V(\mathrm{r}) = \mathrm{A}$$

$$\frac{dU}{d\theta} = \mathrm{A}, \qquad -\mathrm{r}\frac{dV}{dr} = \mathrm{A}$$

$$\mathrm{dU} = \mathrm{A}d\theta, \qquad \mathrm{dV} = -\frac{\mathrm{A}dr}{r}$$

Integrate both sides to obtain U and V.

$$\mathrm{U} = A\int d\theta = \mathrm{A}\theta + c_\theta, \qquad \mathrm{V} = -\mathrm{A}\int\frac{dr}{r} = -\mathrm{A}\ln(\mathrm{r}) + c_r$$

c_θ and c_r are constants of integration

Then

$$\mathrm{f} = \mathrm{A}\theta + c_\theta + \mathrm{i}(-\mathrm{A}\ln\mathrm{r} + c_r)$$

$$= \mathrm{A}(\theta - \mathrm{i}\ln(\mathrm{r})) + (c_\theta + \mathrm{i}c_r)$$

$$= -\mathrm{Ai}(\ln(\mathrm{r}) + \mathrm{i}\theta) + \mathrm{B}, \text{ where } \mathrm{B} = (c_\theta + \mathrm{i}c_r)$$

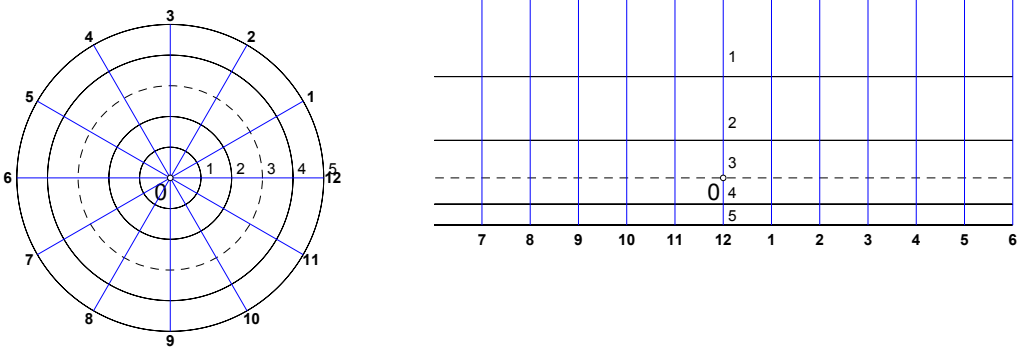
$$= \mathbb{A}\log z + \mathrm{B}, \text{ where } \mathbb{A} = -\mathrm{Ai}$$

As on p. 221, we again have the complex logarithm, which is “uniquely defined (up to constants) as the conformal mapping sending concentric circles to parallel lines.” We show this geometrically with a plot.

To show this geometrically, we set A = -1 (suggested by Vasco) and B = 0. We plot five circles, of which

the third is the unit circle, and 12 rays at angular intervals of Pi/6. Both the unit circle on the LHS and its image on the RHS are plotted with dashed lines. The small numbers indicate the order of the circles and lines. As the angle of the rays increases from zero in (a), the vertical ray images in (b) move farther rightward, and as the angle of the rays decreases from zero in (a), the vertical ray images in (b) move farther leftward. The inner concentric circles have images above the x axis in plot (b) and the outer concentric circles have images below the x axis in plot (b).

Out[5598]=



(a) (b)

Figure 1. f = iALog(z)+B, $\partial_\theta v \equiv 0$