Needham, Chapter 5, Exercise 5, p. 258.
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5. Which of the following are analytic?

(i) $e^{-y}(\cos x + i \sin x)$

Rewrite the expression as $e^{-y}\cos x + i\, e^{-y} \sin x$. We saw from Exercise 3 (i) that an expression of the form $e^{x}\cos y$ is harmonic. More generally, we can guess that expressions of the form $e^{-(\mathrm{Re}(z) \text{ or } \mathrm{Im}(z)) + i(\mathrm{Im}(z) \text{ or } \mathrm{Re}(z))}$, where z = x + iy, are harmonic. From Exercise (ii), we know that both the real and imaginary parts of an analytic function are harmonic. Again, we can guess that e^{-y}(cos x + i sin x) is analytic. We apply the Cauchy-Riemann equations to test this idea:

Let u = $e^{-y}\cos x$ and v = $e^{-y}\sin x$ and apply the Cauchy-Riemann equations:

C-R: $\partial_x u = \partial_y v$

$\partial_x (e^{-y} \cos x) = -e^{-y} \sin x$

$\partial_y (e^{-y} \sin x) = -e^{-y} \sin x$

C-R: $\partial_x v = -\partial_y u$

$\partial_x (e^{-y} \sin x) = e^{-y} \cos x$

$-\partial_y (e^{-y} \cos x) = -(-1)e^{-y} \cos x$

$= e^{-y} \cos x$

$\partial_x u = -e^{-y} \sin x$ and $\partial_y v = -e^{-y} \sin x$, and $\partial_x v = e^{-y}\cos x$ and $-\partial_y u = -\partial_y u$, so the Cauchy-Riemann equations are satisfied. $e^{-y}(\cos x + i \sin x)$ is harmonic. More generally, it appears that $e^{-(\mathrm{Re}(z) \text{ or } \mathrm{Im}(z)) + i(\mathrm{Im}(z) \text{ or } \mathrm{Re}(z))}$ is analytic.

(ii) cos x - i sin y

Rewrite in the form u + iv and apply the Cauchy-Riemann test:

u = cos x, v = sin y

C-R: $\partial_x u = \partial_y v$

$\partial_x u = -\sin x$

$\partial_y v = \cos y$

$\partial_x u$ = –sin x and $\partial_y v$ = cos y, so $\partial_x u \neq \partial_y v$, so the Cauchy-Riemann equations are not satisfied. *cos x - i sin y* is not analytic.

(iii) $r^3 + i3\theta$

This problem is best and most easily solved using the Cartesian-Polar form of the Cauchy-Riemann equations, as Vasco does it, not as I have done it here with the Polar-Polar form.

This function corresponds to $\text{Log}\left(\left(e^{r^3}\, e^{\text{i}3\theta}\right)\right)$, which we can write as $\text{Log}\left(\text{Re}^{\text{i}\Psi}\right)$, where $R = e^{r^3}$ and $\Psi = 3\theta$. We test to see whether $\text{Re}^{\text{i}\Psi}$ is analytic. If it is not, then $\log\left(\text{Re}^{\text{i}\Psi}\right)$ can not be analytic.

Apply the polar-polar form of the Cauchy-Riemann test from page 219.

$$\partial_\theta R = - r R \partial_r \Psi$$

$$\partial_\theta\, e^{r^3} = 0 = - r e^{r^3} \partial_r 3\,\theta$$

and

$$\partial_\theta \Psi = r\, \partial_r R$$

$$\partial_\theta 3\,\theta = r\, \partial_r e^{r^3}$$

$$3 \neq r\, 3\, r^{-2}\, e^{r^3} = 3\, r^{-1}\, e^{r^3}$$

$\partial_\theta R = \partial_\theta\, e^{r^3}$ and $- r R \partial_r \Psi = 0$, and $\partial_\theta \Psi = 3$ and $r\, \partial_r R = 3\, r^{-1}\, e^{r^3}$, so $\partial_\theta \Psi \neq r\, \partial_r R$, so the Cauchy-Riemann equations are not satisfied, so $\text{Re}^{\text{i}\Psi}$ is not analytic. Hence, $\text{Log}(\text{Re}^{\text{i}\Psi})$ is not analytic.

(iv) $\left[r\, e^{r \cos\theta}\right] e^{i(\theta + r\cos\theta)}$

This has the form $\text{Re}^{\text{i}\Psi}$, where $R = r\, e^{r \cos\theta}$ and $\Psi = \theta + r \cos\theta$. We apply the Polar-Polar form of the Cauchy-Riemann equations from page 219.

$$\partial_\theta R = - r R \partial_r \Psi$$

$$\partial_\theta R = \partial_\theta (r\, e^{r \cos\theta}) = r\, r\, (-\sin\theta)\, e^{r \cos\theta}$$

$$= -r^2 e^{r \cos\theta} \sin\theta$$

$$- r R \partial_r \Psi = - r r\, e^{r \cos\theta}\, \partial_r (\theta + r \cos\theta)$$

$$= -r^2 e^{r \cos\theta} \cos\theta$$

$\partial_\theta R = -r^2 e^{r \cos\theta} \sin\theta$ and $- r R \partial_r \Psi = - r r\, e^{r \cos\theta}\, \partial_r (\theta + r \cos\theta)$, so $\partial_\theta R \neq - r R \partial_r \Psi$, so the Cauchy-Riemann

equations are not satisfied. $[r\, e^{r \cos\theta}]\, e^{i(\theta + r\cos\theta)}$ is not analytic.