Needham, Chapter 5, Exercise 4, p. 258.
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4. What is the most general function $u = ax^2 + bxy + cy^2$ that is the real part of an analytic function? Construct this analytic function and express it in terms of z.

Let f = u + iv

If f = u + iv is analytic, then by the Cauchy-Riemann equations,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_x v = -\partial_y u$$

Use the second derivatives of u and use the Laplacian to solve for c.

$$\partial_x u = 2ax + by \qquad \partial_x^2 u = 2a$$

$$\partial_y u = bx + 2cy \qquad \partial_y^2 u = 2c$$

$$2a + 2c = 0, \quad \text{by the Laplacian}$$

$$c = -a$$

Substitute into $\partial_y u$.

$$\partial_y u = bx - 2ay \qquad (1)$$

Then we also have the partials of v.

$$\partial_x v = -\partial_y u = 2ay - bx \qquad (2)$$

$$\partial_y v = \partial_x u = 2ax + by \qquad (3)$$

Rewrite u by substituting for c:

$$u = ax^2 + bxy + cy^2$$

$$= ax^2 + bxy - ay^2$$

$$= a(x^2 - y^2) + bxy \qquad (4)$$

This is the answer to the first part of the question regarding the most general function $u = ax^2 + bxy + cy^2$ that is the real part of an analytic function. We would also like to solve for b, but we don't seem to have another applicable equation. The scalar product goes to zero on both sides of the equation, so there is no multinomial left to work with.

To obtain v, integrate $\partial_y v$ with respect to y:

$$v = \int \partial_y v \, dy + F(x)$$

$$= \int (2ax + by)\, dy + F(x)$$

$$= 2a \int x\, dy + b \int y\, dy + F(x)$$

$$= 2axy + by^2/2 + F(x) \qquad (5)$$

To find F(x), differentiate with respect to x

$$\partial_x v = 2ay + \partial_x F(x)$$

Substituting from (2) above

$$2ay - bx = 2ay + \partial_x F(x)$$

$$\partial_x F(x) = -bx$$

$$F(x) = \int (-bx)\, dx + \text{const.}$$

$$= -bx^2/2 + \text{const.}$$

$$v = 2axy + by^2/2 + F(x) \qquad \text{from (5)}$$

$$= 2axy + by^2/2 - bx^2/2 + \text{const.} \qquad (6)$$

Substitute from (4) and (6) to obtain f = u + iv:

$$f = a(x^2 - y^2) + bxy + i(2axy + by^2/2 - bx^2/2 + \text{const.})$$

$$= a(x + iy)^2 - 2aixy + bxy + 2aixy + biy^2/2 - bix^2/2 + B, \quad \text{where B = i const.}$$

$$= a(x + iy)^2 + bxy + biy^2/2 - bix^2/2 + B$$

$$= a(x + iy)^2 + bxy + (bi/2)\left(y^2 - x^2\right) + B$$

$$= a(x + iy)^2 + bxy - (bi/2)\left(x^2 - y^2\right) + B$$

$$= a(x + iy)^2 + bxy - (bi/2)\left((x + iy)^2 - 2ixy\right) + B$$

$$= az^2 + b\left(xy - (i/2)\left(z^2 - 2ixy\right)\right) + B$$

$= az^2 + b(xy - (i/2)z^2 + (i/2)2ixy) + B$

$= az^2 + b(xy - (i/2)z^2 - xy) + B$

$= az^2 - (ib/2)z^2 + B$

$= (a - ib/2)z^2 + B$

The most general function is $Az^2 + B$.