Needham, Chapter 5, Exercise 31, p. 266.
G. Palmer, June 11, 2016, 4 pm

31 Let C and D be intersecting circles. Let us say that "D is symmetric in C" if reflection (inversion) in C maps D into itself. We know this occurs if and only if D is orthogonal to C, so

D is symmetric in C $\iff$ C is symmetric in D.

Briefly, we may simply say that "C and D are symmetric". Let's see what happens if we generalize C and D to intersecting arcs possessing Schwarz functions and generalize inversion to Schwarzian reflection.

(i) Explain why the statement "D is symmetric in C" is the same as "if the point d lies on D then $R_D[R_C(d)] = \mathcal{R}_C(d)$". Must the arcs be orthogonal?

The equation is equivalent and diagnostic because it only works when d~ is in D, which must be the case if D is symmetric in C. When that is the case, reflection in C sends d to d~, which is also in D.

Let d~ be $\mathcal{R}_C(d)$. (1)

[The tilde over d did not survive the pdf conversion, so we write d~ instead.]

$R_D(d\sim) = d\sim$, by the Schwarzian and because d~ is in D

$R_D[R_C(\mathrm{d})] = R_D(d\sim) = d\sim = \mathcal{R}_C(d)$ (2)

Suppose d~ is not in D. Then (2) would be written:

$R_D[R_C(\mathrm{d})] = R_D(d\sim) = \mathrm{d}^* \neq \mathcal{R}_C(d)$,

where d* is some third point not in D. It is clearly not symmetric.

Must the arcs be orthogonal?

If by "arc" is meant the arc of any smooth curve possessing a Schwarzian reflection, the answer is no, they need not be orthogonal. They need only satisfy the Schwarzian reflection, which entails that reflection in D maps d to itself. The lines must intersect in such a way that Schwarzian reflection can work for D in C and for C in D. This probably means that if they are not perfectly orthogonal, they are nearly orthogonal. Since every Schwarzian function that is not a line or a circle is a little different, an algebraic proof might be difficult. An example of a non-orthogonal intersection would help.

(ii) If D is symmetric in C, deduce that the mappings $(\mathcal{R}_D \circ \mathcal{R}_C)$ and $(\mathcal{R}_C \circ \mathcal{R}_D)$ are equal at points of D.

$\mathcal{R}_D \circ \mathcal{R}_C \equiv \mathcal{R}_D \circ \mathcal{R}_C(\mathrm{d}) = \mathcal{R}_D(\mathrm{d}\sim) = \mathrm{d}\sim$ by (2)

$\mathcal{R}_C \circ \mathcal{R}_D \equiv (\mathcal{R}_C \circ (\mathcal{R}_D(\mathrm{d}) = \mathcal{R}_C(d) = d\sim$, by (1) and (2)

(iii) Using the fact that these two mappings are analytic (why?), deduce that C must also be symmetric in D. thus, as with circles, we may simply say that C and D are symmetric.

The mappings $\mathcal{R}_D \circ \mathcal{R}_C$ and $\mathcal{R}_C \circ \mathcal{R}_D$ are analytic because they are each composed of two anti-conformal mappings (p. 256). By (ii), they are equal at all points of D. Because they are analytic and identical, the mappings must also be equal for all points in C with the same constraints that $R_D(c) = c\sim$, and $c\sim$ in C.

$$\mathcal{R}_D \circ \mathcal{R}_C(c) = \mathcal{R}_D(c) = c\sim = \mathcal{R}_C(c\sim) = \mathcal{R}_C \circ \mathcal{R}_D(c)$$