Needham, Chapter 5, Exercise 30, p. 265.
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30 Let a be a point on a curve K having Schwarz function S(z). By the still unproven result on the infinite differentiability of analytic functions, S(z) may be expanded into a Taylor series in the vicinity of a:

$$S(z) = S(a) + S'(a)(z-a) + \frac{1}{2!} S''(a)(z-a)^2 + \frac{1}{3!} S'''(a)(z-a)^3 + ...$$

(1) Show that the Schwarz function of the tangent line to K at a is given by the first two terms of the series above. This confirms something we saw in [30]: very close to a, reflection in the tangent is a good approximation to Schwarzian reflection.

The Schwarz function of a line through X making angle σ with the real line describes the tangent line at z = a. From Ex. 27, p. 264.

$$S_{L(z)} = ze^{-i2\sigma} + X\left(1 - e^{-i2\sigma}\right)$$

$$S'_L(z) = e^{-i2\sigma}$$

$$S_L^n = 0$$

We substitute these values at into the Taylor series at a:

$$S(a) = \left[ae^{-i2\sigma} + X\left(1 - e^{-i2\sigma}\right)\right] + \left[e^{-i2\sigma}(z-a)\right] + \frac{1}{2!}0(a)(z-a)^2 + \frac{1}{3!}0(z-a)^3 + ...$$

$$= \left[ae^{-i2\sigma} + X\left(1 - e^{-i2\sigma}\right)\right] + [e^{-i2\sigma}(z-a)] + 0 + 0 + ...$$

$$= ae^{-i2\sigma} + X\left(1 - e^{-i2\sigma}\right) + ze^{-i2\sigma} - ae^{-i2\sigma}$$

$$= ze^{-i2\sigma} + X\left(1 - e^{-i2\sigma}\right)$$

Since the second derivative and all higher derivatives go to zero, only the first two terms remain. Their sum is equal to $S_{L(z)}$.

(ii) It is natural to suspect that a better approximation to $\mathcal{R}_K(z)$ would be inversion in the circle of curvature (call it C) of K at α. Let's verify this. Use Ex. 28(ii) and Ex. 27(ii) to find S_C, and show that it may be written as

$$S_{C(z)} = \overline{a} + \frac{2\overline{S}'}{\overline{S}''} + \frac{2(S')^2}{S''}\left[1 - \frac{S''}{2S'}(z-a)\right]^{-1}$$

where it is understood that the derivatives are all evaluated at a. Show that the first three terms of the binomial expansion of S_C agree with those of S, but that they generally differ thereafter. [Hint: You will need the fact that $(\overline{S}'/\overline{S}'') = -(S')^2/S''$ on K. Prove this.]

From ex. 28 (ii) an equation for the center of curvature is

p = {a + 2[S'(a)/S''(a)]}.

Henceforth, we take a as the understood argument.

From ex. 27 (ii)

$$S_C(z) = \overline{p} + \frac{r^2}{z-p}$$

From ex. 28 (i)

$$r^2 = (1/\kappa)^2 = \left(\frac{i}{2}\frac{S''(a)}{[S'(a)]^{3/2}}\right)^{-2} = \left(\frac{2S'^{3/2}}{iS''}\right)^2 = -\frac{4S'^3}{(S'')^2}$$

Substitute for p:

$$S_C(z) = \overline{a + 2[S'/S'']} + \frac{r^2}{z-(a+2[S'/S''])}$$

$$= \overline{a} + 2\frac{\overline{S'}}{\overline{S''}} + \frac{-\frac{4S'^3}{(S'')^2}}{z-(a+2[S'/S''])}$$

$$= \overline{a} + 2\ \frac{\overline{S'}}{S''} - \frac{4\,S'^3}{(S'')^2}\,\frac{1}{(z-(a+2[S'/S'']))}$$

$$= \overline{a} + 2\ \frac{\overline{S'}}{S''} - \frac{4\,S'^3}{(S'')^2}\,\frac{S''}{2\,S'}\,\frac{1}{\frac{S''}{2\,S'}(z-(a+2[S'/S'']))}$$

$$= \overline{a} + \frac{\overline{S'}}{S''} - \frac{2\,S'^2}{S''}\,\frac{1}{\frac{S''}{2\,S'}(z-(a+2[S'/S'']))}$$

$$= \overline{a} + \frac{\overline{S'}}{S''} - \frac{2\,S'^2}{S''}\,\frac{1}{\frac{S''}{2\,S'}(-2[S'/S''])+\frac{S''}{2\,S'}(z-a)\Big)}$$

$$= \overline{a} + \frac{\overline{S'}}{S''} - \frac{2\,S'^2}{S''}\,\frac{1}{\left[-1+\frac{S''}{2\,S'}(z-a)\Big)\right]}$$

$$\overline{a} + \frac{\overline{S'}}{S''} + \frac{2\,S'^2}{S''}\left[1-\frac{S''}{2\,S'}(z-a)\right)\Big]^{-1}$$

QED

Show that the first three terms of the binomial expansion of S_C agree with those of S, but that they generally differ thereafter. [Hint: You will need the fact that $(\overline{S'/S''}) = -(S')^2/S''$. Prove this.]

We learn from Vasco's solution that "the binomial expansion of S_C" actually refers to the result of expanding the expression with square brackets. Do the expansion:

$$(1-z)^{-1} = 1 + z + z^2 + \ldots$$

Let $\alpha = \frac{S''}{2\,S'}$ and substitute α and use the hint:

$$S_C(z) = \overline{a} + \frac{\overline{S'}}{S''} + \frac{2\,S'^2}{S''}\left[1-\frac{S''}{2\,S'}(z-a)\right)\Big]^{-1}$$

$$= \overline{a} + -(S')^2 \Big/ S'' + \frac{2\,S'}{\alpha}\left[1-\alpha\,(z-a)\right)\Big]^{-1}$$

=

$$\overline{a} + -2\,(S')^2 \,/\, (2\,S'\,\alpha) + \frac{S'}{\alpha}\left[1 - \alpha\,(z-a) + \alpha^2 (z-a)^2 + \alpha^2 (z-a)^3 + \ldots\right]$$

$$= \overline{a} - \frac{S'}{\alpha} + \left[\frac{S'}{\alpha} - \frac{S'}{\alpha}\,\alpha(z-a) + \frac{S'}{\alpha}\,\alpha^2 (z-a)^2 + \frac{S'}{\alpha}\,\alpha^3 (z-a)^3 + \ldots\right]$$

$$= \overline{a} - S'\,(z-a) + \alpha\,S'\,(z-a)^2 + \alpha^2\,S'\,(z-a)^3 + \ldots$$

$$= S_C(a) - S'\,(z-a) + \frac{S''}{2\,S'}\,S'\,(z-a)^2 + \left(\frac{S''}{2\,S'}\right)^2 S'\,(z-a)^3 + \ldots$$

$$= S_C(a) - S'\,(z-a) + \frac{S''}{2}\,(z-a)^2 + \frac{(S'')^2}{4\,S'}\,(z-a)^3 + \ldots$$

The first three terms of both series are

$$S_C(z) - S'\,(z-a) + \frac{S''}{2}\,(z-a)^2$$

Prove that $\overline{S'/S''} = -(S')^2/S''$.

From Ex. 28, we have:

$$S' = e^{-2\,i\phi}, \quad S'' = dS'/dz = -i2e^{-i2\phi}\,d\phi/dz, \quad \kappa \equiv \dot{\phi} = \frac{d\phi}{dl}, \quad dz = e^{i\phi}\,dl$$

First, find $\frac{S''}{\overline{S''}}$, which will be needed later.

$$S'' = -i2e^{-i2\phi}\,\frac{d\phi}{dl}\,\frac{dl}{dz} = = -i2e^{-i2\phi}\,\frac{d\phi}{dl}\,\frac{dl}{e^{i\phi}\,dl} = -i2e^{-3\,\phi}\,\frac{d\phi}{dl} = -i2e^{-3\,\phi}\,\kappa$$

$$\overline{S''} = i2e^{3\,\phi}\,\kappa \quad (\kappa \text{ is real, so } \overline{\kappa} = \kappa)$$

$$\frac{S''}{\overline{S''}} = \frac{-i2e^{-3\,\phi}\,\kappa}{i2e^{3\,\phi}\,\kappa} = -e^{-i6\phi} \tag{1}$$

Now we have enough to prove that $\overline{S'/S''} = -(S')^2/S''$.

$$\overline{S'/S''} = \overline{e^{-i2\phi}\,/\,S''} = \frac{e^{i2\phi}}{\overline{S''}} = \frac{e^{i2\phi}}{\overline{S''}}\,\frac{S''}{S''} = \frac{e^{i2\phi}\left(S''/\overline{S''}\right)}{S''} \tag{2}$$

$= \frac{e^{i2\phi}\left(-e^{-i6\phi}\right)}{S''} = \frac{-e^{-i4\phi}}{S''} = \frac{-(S')^2}{S''}$ by substituting (1) into parens. in (2)

(iii) If the curvature κ of K were constant then K would be identical to its circle of curvature. The fact that S and S_C disagree beyond the third term thus reflects the fact that κ does change. One is thus led to guess that the faster κ changes,the greater the discrepancy between $\mathcal{R}_K$ and inversion in C. Continuing from the last part, use Ex. 28(iii) to verify this hunch in the following precise form:

$$S_C(z) - S(z) \approx (i/3)[S']^2 \dot{\kappa} (z-a)^3$$

The result of Ex. 28 (iii) is $\dot{\kappa} = \frac{i}{2S'}\{S(z),z\}$.

Subtract S(z) series from the $S_C(z)$ series calculated above and compare the result to (3)

$$S_C(z) - S(z) = S_C(a) - S_C'(a)(z-a) + \frac{S_C''(a)}{2}(z-a)^2 + \frac{(S_C'')^2(a)}{4}(z-a)^3 + \ldots$$

$$- (S(a) + S'(a)(z-a) + \frac{1}{2!}S''(a)(z-a)^2 + \frac{1}{3!}S'''(a)(z-a)^3 + \ldots)$$

Treat the first three terms as equal and compare the difference of the fourth terms. It appears that one component of the result resembles and approximates the second form of the Schwarzian derivative.

$$S_C(z) - S(z) \approx (\frac{(S_C'')^2}{4S'}(z-a)^3 + \ldots) - (\frac{1}{3!}S'''(z-a)^3 + \ldots)$$

($S_C(a)$ and $S(a)$ are implied)

$$= (\frac{(S_C'')^2}{4S'} - \frac{S'''}{3!})(z-a)^3$$

$$= -(\frac{S'''}{3!} - \frac{(S_C'')^2}{4S'})(z-a)^3$$

$$= -\frac{1}{6}S'(\frac{6S'''}{3!S'} - \frac{6(S_C'')^2}{4(S')^2})(z-a)^3$$

$$= -\frac{1}{6} S'\left(\frac{S'''}{S'} - \frac{3}{2}\left(\frac{S_C''}{S'}\right)^2\right)(z-a)^3$$

$$= -\frac{1}{6} S'\{S(z),z\}(z-a)^3$$

From 28(iii), we have $\dot{\kappa} = \frac{i}{2S'}\{S(z),z\}$, so $\{S(z),z\} = 2S'\dot{\kappa}/i = -i2S'\dot{\kappa}$. Then

$$S_C(z) - S(z) \approx -\frac{1}{6} S'(-i2S'\dot{\kappa})(z-a)^3$$

$$= \frac{i}{3} (S')^2 \dot{\kappa}(z-a)^3$$

QED

This verifies the hunch that *the faster κ changes,the greater the discrepancy between $\mathcal{R}_K$ and inversion in C*.

Question:

How do we know that the equation is precise? What is the rationale for discarding the n > 4 terms? Is it because the coefficients become smaller and the Schwarzian converges? For a general smooth curve K, it can't be because derivatives go to zero after the fourth term. How do we know that the n > 4 terms become sufficiently small that they make an insignificant contribution to the error?