Needham, Chapter 5, Exercise 29, p. 265 (and 264)
G. Palmer, June 2, 2016

29 Check the result of Ex. 28 (i) by applying the results of Ex. 27.

In Ex. 27 (i), L is a line, so it should have zero curvature. Then we are checking to see if

$$\left(\kappa \equiv \dot{\phi} = \frac{i}{2}\frac{S''(a)}{[S'(a)]^{3/2}}\right) = 0$$

In order for that to be true, S''(a) must be zero where

$$\mathcal{S}_L(z) = z\,e^{-i2\sigma} + \mathcal{X}\left(1 - e^{-i2\sigma}\right)$$

Since L is a line, we take σ to be a constant. Then

$$(S_L(z))' = \left(ze^{-i2\sigma} + \mathcal{X}\left(1 - e^{-i2\sigma}\right)\right)'$$

$$= e^{-i2\sigma} + 0 = e^{-i2\sigma}$$

$$(S_L(z))'' = 0$$

QED

Deduce that $|\kappa| = |S''/2|$

$\kappa = 0$, so $|\kappa| = 0 = \left|\frac{0}{2}\right| = \left|\frac{(S_L(z))''}{2}\right|$

In Ex. 27(ii), C is a circle with centre p and radius r. We are to show that it's Schwarz function is

$$S_C(z) = \overline{p} + \frac{r^2}{z-p}$$

A circle has constant curvature of 1/r. Then we should find that

$$\left(\kappa \equiv \dot{\phi} = \frac{i}{2}\frac{S''(a)}{[S'(a)]^{3/2}}\right) = \frac{1}{r}$$

$$(S_C(z))' = 0 + r^2\left((z-p)^{-1}\right)'$$

$$= -r^2(z-p)^{-2}$$

$$((S_C(z))')^{3/2} = \left(-r^2(z-p)^{-2}\right)^{3/2} = (-1)^{3/2}\left(-r^2\right)^{3/2}\left((z-p)^{-2}\right)^{3/2}$$

$$= \pm i r^3 (z-p)^{-3}$$

$$(S_C(z))'' = 2r^2 (z-p)^{-3}$$

$$\kappa = \pm \frac{i}{2} \frac{2r^2 (z-p)^{-3}}{i r^3 (z-p)^{-3}} = \pm \frac{1}{r}$$

QED

Deduce that $|\kappa| = |S''/2|$.

$|\kappa| = \left|\frac{2r^2 (z-1)^{-3}}{2}\right| = \left| r^2 \, (z-p)^{-3} \right| = \left| \frac{r^2}{z^3} \right|$, because p is the centre of C

$= \left| \frac{1}{r} \right| = \left| \pm \frac{1}{r} \right|$