Needham, Chapter 5, Exercise 28, p. 265.
G. Palmer, June 1, 2016

28 Let a be a point on a (directed) curve K having Schwarz function S(z).

(i) Show that the curvature of K at a is

$$\kappa \equiv \dot{\phi} = \frac{i}{2} \frac{S''(a)}{[S'(a)]^{3/2}},$$

where ϕ is the angle in [30], and the dot denotes differentiation with respect to distance / along K (in the given sense).

With the hints, it is given that $\dot{\phi} = dl/d\phi$, S'' = dS'/dz, dz = $e^{i\phi}$dl, and S' = $e^{-i2\phi}$.

Calculate S'' and $[S'(a)]^{3/2}$. Then try to reduce the RHS to an equation containing $\frac{d\phi}{dl}$ because $\dot{\phi} = \frac{d\phi}{dl}$.

$$S'' = dS'/dz = -i2e^{-i2\phi}d\phi/dz$$

$$[S'(a)]^{3/2} = \left(e^{-i2\phi}\right)^{3/2} = e^{-i3\phi}$$

$$\frac{i}{2} \frac{S''(a)}{[S'(a)]^{3/2}} = \frac{i}{2} \frac{-i2e^{-i2\phi}\, d\phi/dz}{e^{-i3\phi}} = e^{i\phi} \frac{d\phi}{dz}$$

$$= e^{i\phi}\left(d\phi \Big/ \left(e^{i\phi}\, dl\right)\right) = \frac{d\phi}{dl} = \dot{\phi} \equiv \kappa$$

Deduce that

$$|\kappa| = |S''/2|$$

$$|S''/2| = |(-i2e^{-i2\phi}d\phi/dz)/2|$$

$$= |d\phi/dz|$$

$$= |d\phi \big/ e^{i\phi}dl|$$

$$= |d\phi/dl| = |\dot{\phi}| = |\kappa|$$

(ii) Deduce that the centre of curvature of K at a is {a + 2[S'(a)/S''(a)]}.

By [30], ϕ is the angle between the tangent and the real line. Let q be the center of the circle of curvature. We can write

$$a = q - \left| a - q \right| ie^{i\phi}$$ (minus sign because $ie^{i\phi}$ points at q from a)

$$q = a + rie^{i\phi}$$

$$= a + \frac{1}{k} i e^{i\phi}$$

$$= a + i[S'(a)]^{\frac{-1}{2}} \Big/ \left(\frac{i}{2}\left((S''(a)) \Big/ [S'(a)]^{3/2}\right)\right) \quad \text{(see (iii) for derivations)}$$

$$= a + \frac{2\,S'(a)}{S''(a)}$$

(iii) Show that the rate of change of the curvature of K is given by the "Schwarzian derivative" [Ex. 19] of the Schwarz function:

$$\dot{\kappa} = \frac{1}{2\,S'} \{S(z), z\}.$$

The key is to recognize that $\dot{\kappa}$ is $\frac{d\kappa}{dl}$, not $\frac{d\kappa}{dz}$.

$$\kappa = \frac{i}{2} \frac{S''}{[S']^{3/2}}, \quad \text{from (i)}$$

$$\frac{d\kappa}{dz} = \frac{i}{2} \left(S'' \Big/ [S']^{3/2}\right)'$$

$$\frac{d\kappa}{dl}\frac{dl}{dz} = \frac{i}{2} (S'' [S']^{-3/2})'$$

$$\frac{d\kappa}{dl} = \frac{i}{2}\frac{dz}{dl} [(S'')' [S']^{-3/2} + S'' ([S']^{-3/2})']$$

$$S'(a) = e^{-i2\phi}, \text{ given}$$

$$S'(a)^{1/2} = e^{-i\phi}$$

$$e^{i\phi} = S'(a)^{-1/2}$$

$$\dot{\kappa} = \frac{i}{2} e^{i\phi} [(S'')' [S']^{-3/2} - (3/2)\, S'' [S']^{-5/2} S''], \text{ because } \frac{dz}{dl} = e^{i\phi}$$

$$= \frac{i}{2} S'^{(-1/2)} [(S'')' \Big/ [S']^{3/2} - (3/2)(S'')^2 \Big/ [S']^{5/2}]$$

$$= (i S'^{(-1/2)} \Big/ 2\,(S')^{1/2}) [\frac{S'''}{S'} - (3/2) \left(\frac{S''}{S'}\right)^2]$$